J. Fluid Mech. (1993), *vol.* 254, *pp.* 183–211

Interactions between a solid surface and a viscous compressible flow field

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(Received 3 October 1988 and in revised form 22 February 1993)

This paper presents a general theory and physical interpretation of the interaction between a solid body and a Newtonian fluid flow in terms of the vorticity $\boldsymbol{\omega}$ and the compression/expansion variable Π instead of primitive variables, i.e. velocity and pressure. Previous results are included as special simplified cases of the theory. The first part of this paper shows that the action of a solid wall on a fluid can be exclusively attributed to the creation of a vorticity-compressing $\boldsymbol{\omega}$–Π field directly from the wall, a process represented by respective boundary fluxes. The general formulae for these fluxes, applicable to any Newtonian flow over an arbitrarily curved surface, are derived from the force balance on the wall. This part of the study reconfirms and extends Lighthill's (1963) assertion on vorticity-creation physics, clarifies some currently controversial issues, and provides a sound basis for the formulation of initial boundary conditions for the $\boldsymbol{\omega}$–Π variables.

The second part of this paper shows that the reaction of a Newtonian flow to a solid body can also be exclusively attributed to that of the $\boldsymbol{\omega}$–Π field created. In particular, the integrated force and moment formulae can be expressed solely in terms of the boundary vorticity flux. This implies an inherent unity of the action and reaction between a solid body and a $\boldsymbol{\omega}$–Π field.

In both action and reaction phases the $\boldsymbol{\omega}$–Π coupling on the wall plays an essential role. Thus, once a solid wall is introduced into a flow, any theory that treats $\boldsymbol{\omega}$ and Π separately will be physically incomplete.

1. Introduction

Since the influential works of Truesdell (1954), Lighthill (1963), Batchelor (1967), and Chorin (1973) it has been found instructive to interpret the dynamic interactions inside a fluid or between a fluid and a solid surface in terms of vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ rather than with primitive variables $\boldsymbol{u}$ and p. Meanwhile, various vorticity-based formulations of flow problems and corresponding numerical methods have become an important complement to the usual $\boldsymbol{u}$–p computation. The vitality of the vorticity interpretation and computation relies on the fundamental difference between a fluid and a solid. The former cannot withstand shearing, as precisely represented by vorticity. Moreover, as the skew-symmetric part of the velocity gradient, a knowledge of vorticity implies knowing not only the fluid motion at a single spatial point, but also the relation of that motion with those of neighbouring points. Thus, the vorticity reflects the dynamic mechanism of the shearing process more directly than does velocity.

However, in spite of the above pioneering works and other subsequent research, vorticity interpretation and computation are still unsatisfactory when a solid wall occurs in the flow field. Understanding and simulating the vorticity creation process at a wall remains unclear, especially in its full general form. From the early discussions of Lighthill and Batchelor, and also of Morton (1984), one may find some fundamental clues to such a creation process, but not a complete theory. Hence, this problem is still considered an extremely important unanswered question in fluid mechanics (Trefethen & Panton 1990; Majda 1987; Hald 1991). Sherman (1990) noted the problem by saying, 'The irony of the Random-Vortex Method is that it deals entirely with vorticity, but does not readily yield a good estimation of the value of ω at the wall'.

A thorough resolution of all the problems associated with vorticity creation from the wall must be, and can only be, achieved by a rigorous and sufficiently general theoretical analysis of this process. A key step is to fully recognize the importance of coupling at a wall between the shearing process and the longitudinal compression/expansion process (or compressing process, for short). Dealing entirely with vorticity can by no means cover everything. In fact, on a surface element $\delta \boldsymbol{S} = \boldsymbol{n}\delta S$ of a Newtonian fluid the stress $\boldsymbol{t}$ has three constituents (Wu & Wu 1992):

$$\boldsymbol{t} = -\Pi \boldsymbol{n} + \boldsymbol{\tau} + \boldsymbol{t}_s, \tag{1.1}$$

where

$$\Pi = p - (\lambda + 2\mu)\vartheta \tag{1.2}$$

is the variable describing the normal compressing process with $\vartheta = \nabla \cdot \boldsymbol{u}$ being the dilatation and μ, λ the first and second viscosities;

$$\boldsymbol{\tau} = \mu \boldsymbol{\omega} \times \boldsymbol{n} \tag{1.3}$$

is the tangential shear stress; and

$$\boldsymbol{t}_s \delta S = -2\mu \boldsymbol{B} \cdot \delta \boldsymbol{S} = 2\mu \frac{\mathrm{D}}{\mathrm{D}t}(\delta \boldsymbol{S}), \quad \boldsymbol{B} \equiv \vartheta \boldsymbol{I} - \nabla \boldsymbol{u}, \tag{1.4}$$

is the viscous resistance to surface deformation, with $\boldsymbol{I}$ being the unit tensor. Thus far the variable Π has not been given a definite name. In this paper it will be called the 'compressing variable' (or more precisely, the compression/expansion variable). The tensor $\boldsymbol{B}$ in (1.4) is the surface deformation tensor. Wu & Wu (1992) showed that while $\boldsymbol{t}_s$ disappears on a rigid wall and in many commonly encountered situations, the normal stress $-\Pi \boldsymbol{n}$ is nevertheless as important as the shearing stress $\mu\boldsymbol{\omega} \times \boldsymbol{n}$. As a result, there is a strong $\boldsymbol{\omega}$–Π coupling on solid surfaces due to the adherence condition. This fact can be seen most clearly from an observation by Truesdell (1954). If μ is constant, then the body force (external force plus inertial force) of a fluid element per unit volume has a natural Stokes–Helmholtz decomposition

$$\rho(\boldsymbol{f} - \boldsymbol{a}) = \nabla \Pi + \nabla \times (\mu \boldsymbol{\omega}), \tag{1.5}$$

where $\boldsymbol{f}$ is the external body force per unit mass, ρ is density and $\boldsymbol{a} = \mathrm{D}\boldsymbol{u}/\mathrm{D}t$ is the acceleration. Therefore, the uniquely defined Π and $\mu\boldsymbol{\omega}$ are just a pair of Stokes–Helmholtz potentials of the body force. The mutual coupling of Stokes–Helmholtz scalar and vector potentials has also been studied by Wu & Wu (1992).

The above discussion indicates that, as a rational complement to the usual $\boldsymbol{u}$–p approach, the interpretation and formulation of fluid dynamics in terms of $\boldsymbol{\omega}$ alone should be extended to that in terms of both $\boldsymbol{\omega}$ and Π. Here 'interpretation' is used to mean a point of view of looking at things, without computing relevant quantities. For example, $\boldsymbol{\omega}$ and Π can be either inferred from primitive variables or directly solved from the derived equations for $\boldsymbol{\omega}$ and Π by taking the curl and divergence of (1.5). On the other hand, 'formulation' is used to mean a well-posed problem for solving relevant variables. Thus, while the $\boldsymbol{u}$–p formulation aims at solving the Navier–Stokes equation directly, the $\boldsymbol{\omega}$–Π formulation should provide a basis for solving the initial boundary value problems of the coupled $\boldsymbol{\omega}$–Π equations.

Since this paper aims at the general theory of the ω–Π interpretation, no specific flow problem is to be solved here. The ω–Π formulation, along with relevant numerical methods and examples, is the topic of a companion paper in preparation (for a concise version, see Wu *et al.* 1992). The study here, therefore, does not need to use the coupled ω–Π equations at all. Rather, it is based on the general Navier–Stokes equation with variable viscosities, which can be written as a slight extension of (1.5):

$$\rho(\boldsymbol{f}-\boldsymbol{a})-2\boldsymbol{B}\cdot\nabla\mu = \nabla\Pi+\nabla\times(\mu\boldsymbol{\omega}), \tag{1.6}$$

where $\boldsymbol{B}$ is the surface deformation tensor. The usual form is the negative of (1.6). The extra term due to $\nabla\mu$ can be further decomposed if one wishes. However, this decomposition is far less natural than (1.5) and involves integral operators, which will not be addressed here.

More specifically, this paper concentrates on the action of a solid surface on the ω–Π field and the reaction of the ω–Π field created to the solid body. As previously seen, this interaction stands at the centre of the ω–Π interpretation of fluid dynamics. Therefore, this paper first provides a careful reconfirmation of Lighthill's assertion that all vorticity created at a solid wall, including the vorticity right on the wall and a possible initial vortex sheet, is exclusively due to the boundary vorticity flux (which he called the 'vorticity source strength'). A set of general formulae is presented for this flux and the 'Π flux' that occur at arbitrarily curved solid walls in three-dimensional, compressible viscous flows. These formulae express the force balance on the wall and improve one's understanding of the relevant physics.

The second contribution of this paper is a series of new integrated force and moment formulae using either the ω–Π distribution or their boundary fluxes, reflecting the reaction of the ω–Π field created to the solid body. The key result is the establishment of force and moment formulae in terms of boundary vorticity flux. These formulae further justify the analysis based on the vorticity creation. Hence they clearly reveal the inherent unity of the mechanism by which the ω–Π field is generated and coupled at the solid wall and that by which the generated ω–Π field reacts to the body.

In both action and reaction phases, ω–Π coupling at a solid wall through the adherence condition plays a crucial role. In fact, it is this coupling phenomenon that makes the fluid dynamic interaction rich and colourful. Lighthill (1963) first discussed the underlying physics of this coupling in the context of vorticity creation. Since then he has restressed its importance several times (Lighthill 1979, 1986*a*, *b*). Coupling also leads to some interesting results in the reaction phase, shedding further light on the physics of the interaction. Moreover, coupling provides proper boundary conditions for the ω–Π formulation, with the boundary behaviour of shearing and compressing processes as the main and direct concern (Wu *et al.* 1992).

This paper is arranged as follows. Part 1 (§§2–4) deals with the action phase. It starts with the physical concept and source of boundary vorticity, followed by the general formulae for boundary ω–Π fluxes. The study then turns to a more detailed analyses of the physics. Some currently controversial issues are clarified. Part 2 (§§5–7) addresses the reaction phase. The general physical picture is reviewed first, then the new force and moment formulae based solely on boundary vorticity flux are derived. The existence or non-existence of parallel formulae in terms of the compressing variable Π and its flux is explored. Finally, some concluding remarks are presented.

Part 1. ω–Π production from a solid surface

This part discusses the mechanism of ω–Π production from a solid surface (the action phase), forming the basis of the entire analysis. The emphasis is on the vectorial shearing process, which is much more complicated than the scalar compressing process. Once the former is clarified, the latter follows easily.

2. The concept and source of boundary vorticity

Four quantities are involved in the interaction of a fluid and the solid surface ∂B: the boundary vorticity ω_B, normal stress Π_B and their respective normal gradients as denoted by

$$\boldsymbol{\sigma} = \boldsymbol{n}\cdot\nabla(\mu\boldsymbol{\omega}) \quad \text{and} \quad \delta = -\boldsymbol{n}\cdot\nabla\Pi \quad \text{at} \quad \partial B, \tag{2.1 a, b}$$

where $\boldsymbol{n} = -\hat{\boldsymbol{n}}$ is the unit normal vector directed out of the fluid (subscript B is used to denote *fluid* quantities at solid surface ∂B. This subscript will be omitted if no confusion is caused). $\boldsymbol{\sigma}$ is called the *boundary vorticity flux*. For convenience δ is called the *boundary compressing flux*, although it should be more properly called the boundary normal force. These four quantities, $\boldsymbol{\omega}_B$, Π_B, $\boldsymbol{\sigma}$ and δ, constitute a sufficient boundary dynamic information set. Once their distribution over ∂B is known at any time t, the whole flow problem can be solved by either the Neumann or Dirichlet formulation (Wu *et al.* 1992). Evidently, among these four quantities there exist some dynamic relations which reveal the ω–Π production mechanism. Thus, the task is to obtain all relations in their most general forms, guided by unambiguous physical arguments.

First, the concept 'boundary vorticity', and even that of 'vorticity' itself, needs to be clarified. According to §1, the interpretation of vorticity should be based on a deformable fluid element (rather than a point-like particle) in which the motion of neighbouring points relative to the element centre can be analysed. Stokes interpreted vorticity as twice the angular velocity of such a fluid element. The angular velocity in a fluid can then be interpreted in terms of circulation (Batchelor 1967). But a fluid element may have circulation only if it belongs to open subsets of the fluid domain V, which does not include the solid surface ∂B. However, a small hemispheric fluid element with its centrepoint at ∂B is conceivable (figure 1 *a*). Although the adherence condition prohibits this element from rotating or translating relative to ∂B, the element does experience shearing when there is a momentum flow over its top. It is the shearing that rotates the principal axes of the strain-rate tensor at the centre of the element and gives it the boundary vorticity $\boldsymbol{\omega}_B$.

To quantify the above, a remarkable result discovered by Caswell (1967) concerning boundary vorticity $\boldsymbol{\omega}_B$ and boundary dilatation ϑ_B is first used. Caswell proved that for any continuous medium the strain-rate tensor $\boldsymbol{D}$ on a stationary boundary ∂B, with an adherence condition being imposed, may be expressed solely by $\boldsymbol{\omega}_B$ and ϑ_B. While Caswell's formula was confined to a stationary wall, it can be easily extended to a moving wall with angular velocity $\boldsymbol{W}$:

$$2\boldsymbol{D} = 2\boldsymbol{n}\boldsymbol{n}\vartheta + \boldsymbol{n}(\boldsymbol{\omega}'\times\boldsymbol{n}) + (\boldsymbol{\omega}'\times\boldsymbol{n})\boldsymbol{n} \quad \text{on} \quad \partial B, \tag{2.2}$$

where

$$\boldsymbol{\omega}' = \boldsymbol{\omega} - 2\boldsymbol{W} \tag{2.3}$$

is the relative vorticity, satisfying the boundary condition

$$\boldsymbol{\omega}'\cdot\boldsymbol{n} = 0 \quad \text{or} \quad \boldsymbol{\omega}\cdot\boldsymbol{n} = 2\boldsymbol{W}\cdot\boldsymbol{n} \quad \text{on} \quad \partial B, \tag{2.4}$$

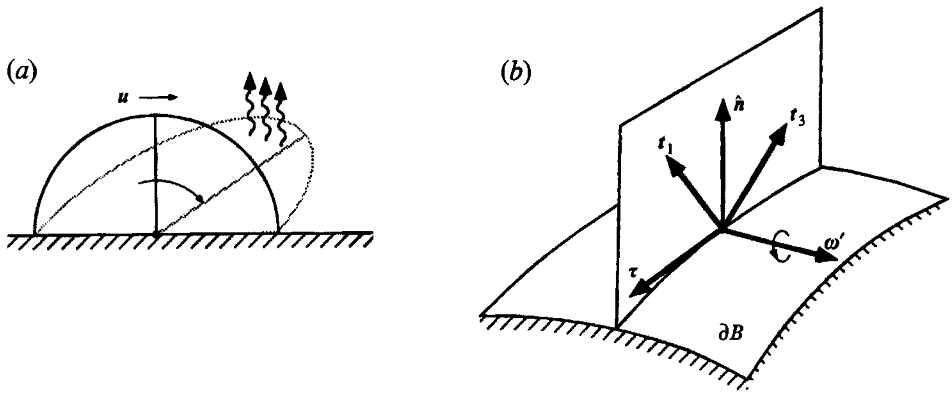

FIGURE 1. The physics of boundary vorticity. (*a*) A fluid element sticking to the wall. (*b*) The principal axes $\boldsymbol{t}_1$ and $\boldsymbol{t}_3$ rotate around $\boldsymbol{e}_2 = \boldsymbol{\omega}'/|\boldsymbol{\omega}'|$. This boundary vorticity is transferred to the interior of the fluid by diffusion.

which is a consequence of the no-slip condition. Therefore, for a Newtonian fluid with the stress tensor $\boldsymbol{T} = (-p+\lambda\vartheta)\boldsymbol{I}+2\mu\boldsymbol{D}$, the wall stress is (compare with (1.1) to (1.4))

$$\boldsymbol{t} = \boldsymbol{T}\cdot\hat{\boldsymbol{n}} = -\Pi\hat{\boldsymbol{n}}+\mu\boldsymbol{\omega}'\times\hat{\boldsymbol{n}}. \tag{2.5}$$

Because of (2.4), the right-hand sides of (2.2) and (2.5) have only three independent component variables. It then follows that $\boldsymbol{\omega}'_B$ is exactly equivalent to the wall shear stress $\boldsymbol{\tau}_B$, a well-known result:

$$\boldsymbol{\tau}_B = \mu\boldsymbol{\omega}'_B\times\hat{\boldsymbol{n}} \quad \text{or} \quad \mu\boldsymbol{\omega}'_B = \hat{\boldsymbol{n}}\times\boldsymbol{\tau}_B. \tag{2.6}$$

Usually $\boldsymbol{\tau}_B$ is defined on ∂B only, but once expressed by $\boldsymbol{\omega}$ it can be conveniently 'continued' into V.

Now, let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \hat{\boldsymbol{n}})$ be unit orthogonal base vectors on the wall, such that $\boldsymbol{\tau}_B = \tau\boldsymbol{e}_1$ and $\boldsymbol{\omega}'_B = \omega'\boldsymbol{e}_2$. Then from (2.2) it can be easily shown that, for any fluid flow over a rigid wall, the unit vectors $\boldsymbol{t}_i (i = 1,2,3)$ of the principal axes of the strain-rate tensor are given by

$$\left.\begin{aligned} \boldsymbol{t}_1 &= (f_1(\vartheta/\omega'), 0, f_2(\vartheta/\omega')), \\ \boldsymbol{t}_2 &= \boldsymbol{e}_2, \\ \boldsymbol{t}_3 &= (-f_2(\vartheta/\omega'), 0, f_1(\vartheta/\omega')), \end{aligned}\right\} \tag{2.7a}$$

where
$$f_{1,2}\left(\frac{\vartheta}{\omega'}\right) = \frac{1}{\sqrt{2}}\left(1 \mp \frac{\vartheta/\omega'}{(1+(\vartheta/\omega')^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}} \tag{2.7b}$$

(see figure 1*b*). Thus, for incompressible flows the angle between $\boldsymbol{t}_1$ and $\boldsymbol{\tau}_B$ is always $\frac{1}{4}\pi$ (a result well recognized for the average inclination angle of hairpin vortices in turbulent boundary layers). The $\boldsymbol{t}_1, \boldsymbol{t}_3$ axes, the strongest stretching and shrinking directions of the hemisphere respectively, rotate around $\boldsymbol{e}_2$ with an angular velocity $\frac{1}{2}\omega'$.

The above observation suggests that the only consistent interpretation of vorticity, applicable to fluid elements both in the interior of fluid and on a solid boundary, is twice the angular velocity of the principal axes of the strain-rate tensor. According to Truesdell (1954), this interpretation was due to Boussinesq.

Now turn to the central problem of how vorticity is created at a solid surface. Although the hemisphere in figure 1(*a*) cannot move relative to the wall, its vorticity,

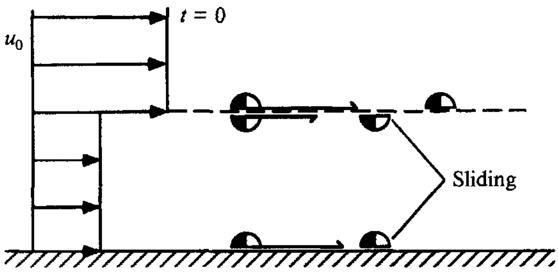

FIGURE 2. Tangential discontinuity of velocity in inviscid flow.

once created, can be diffused into the flow field along with the momentum diffusion. The vorticity can then be convected. Thus, for incompressible flow without a non-conservative external body force, all the interior vorticity must come from the diffusion of $\boldsymbol{\omega}_B$ exclusively. In order to understand the vorticity creation process, therefore, one only needs to find the source of boundary vorticity $\boldsymbol{\omega}_B$. While the general result is given in §3, it is sufficient here to consider a unidirectional flow over a flat plate at $y = 0$, with $\boldsymbol{u} = (u(y, t), 0, 0)$, $\boldsymbol{\omega} = (0, 0, \omega(y, t))$ and $\rho = 1$. Assume that the fluid and the plate are at rest for $t < 0$, and at $t = 0$ there suddenly appear a uniform fluid velocity $\boldsymbol{U} = (U, 0, 0)$, a tangential motion of the plate with speed $b(t)$, and a uniform, time-dependent tangential pressure gradient $\partial p/\partial x = P(t)$. The question is what happens to those fluid hemispheres centred at the plate.

First, suppose that the fluid is restrictively inviscid. Then all fluid elements, including those hemispheres, will simply slide over the plate (figure 2). A discontinuity of velocity $U - b(t)$ appears between the fluid and the plate (with different initial conditions, a similar discontinuity may occur inside the fluid as well). This discontinuity in no way represents a vortex sheet within which $|\boldsymbol{\omega}_B| = \infty$ because the principal axes of the strain-rate tensor of those fluid hemispheres on the wall do not rotate. It is this almost trivial point that is quite controversial. Some authors identify such an inviscid discontinuity as a vortex sheet, leading to the incorrect conclusion that the vorticity creation is a purely kinematic process and has nothing to do with the no-slip condition (Morton 1984; Morino 1986, 1990). In fact, Lagerstrom (1973) made a clear distinction between an Euler solution and the Euler limit of a Navier–Stokes solution as $\nu \to 0$. For the latter the no-slip condition is satisfied, causing a vortex sheet on the wall. Unlike the pure sliding case shown in figure 2, within a true vortex sheet the principal axes of the strain-rate tensor rotate with infinite angular velocity. This happens, ideally, either in the limit of vanishing viscosity or before the diffusion smooths out a sudden tangential change of flow conditions.

This being the case, one should introduce the viscosity and no-slip condition into this unidirectional flow. It then follows that the vorticity distribution is given by

$$\omega(y, t) = \int_0^t \frac{\sigma(\tau)}{[\pi\nu(t-\tau)]^{\frac{1}{2}}} \exp\left[-\frac{y^2}{4\nu(t-\tau)}\right] \mathrm{d}\tau \tag{2.8}$$

(a special case of (4.8)), where $\sigma = -\nu(\partial\omega/\partial y)_B$ is the boundary vorticity flux. This flux is the result of a balance of tangential forces on the wall. Applying the Navier–Stokes equation to the wall and using the no-slip condition gives

$$\sigma(t) = \frac{\mathrm{d}b}{\mathrm{d}t} + P(t), \tag{2.9}$$

implying that the tangential component of viscous force, σ, on the volumetric fluid element (now the hemispheres) resists inertial and pressure forces. It is this resistance that make the fluid hemispheres on the wall experience a definite torque, whose time-accumulated effect in turn makes the wall experience a tangential stress $-\nu\omega_B$. Indeed, at $y = 0$ (2.8) gives

$$\omega_B(t) = \frac{1}{(\pi\nu)^{\frac{1}{2}}}\int_0^t \frac{\sigma(\tau)}{(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}\tau, \tag{2.10}$$

indicating clearly that the boundary vorticity is created exclusively by the force balance over the hemispheres, at least for the present model flow. Moreover, (2.8) also yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \omega(y,t)\,\mathrm{d}y = \sigma(t).$$

Thus Lighthill (1963) correctly called $\boldsymbol{\sigma}$ the vorticity source strength (per unit area per unit time). However, by (2.1*a*) it appears as a diffusive flux.

The flux $\boldsymbol{\sigma}$ can be singular as well as regular. In the present example the impulsively started flow U must come from an impulsive pressure gradient, and the impulsively started wall motion must cause an impulsive inertial force. Thus,

$$P(t) = -U\delta(t) \quad \text{and} \quad \frac{\mathrm{d}b}{\mathrm{d}t} = b_0\,\delta(t) \quad \text{for} \quad 0^- \leqslant t \leqslant 0^+,$$

where $\delta(t)$ is the delta function. That is, a hemisphere sticking to the wall will experience an impulsive force $-(U-b_0)\,\delta(t) = \sigma(t)$ for $0^- \leqslant t \leqslant 0^+$. After separating this singular part from the rest, (2.8) and (2.10) become

$$\omega(y,t) = \frac{\gamma_0}{(\pi\nu t)^{\frac{1}{2}}}\exp\left(-\frac{y^2}{4\nu t}\right) + \int_{0^+}^t \frac{\sigma(\tau)}{[\pi\nu(t-\tau)]^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4\nu(t-\tau)}\right]\mathrm{d}\tau, \tag{2.11}$$

$$\omega_B(t) = \frac{\gamma_0}{(\pi\nu t)^{\frac{1}{2}}} + \frac{1}{(\pi\nu)^{\frac{1}{2}}}\int_{0^+}^t \frac{\sigma(\tau)}{(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}\tau, \tag{2.12}$$

where $\gamma_0 = -(U-b_0)$ is precisely the strength of the initial vortex sheet. It is therefore evident that, contrary to the opinion of Morton and Morino, this initial vortex sheet is also a consequence of the tangential force balance on the wall under a no-slip condition. As shown by (2.11), once this singular part is separated, in numerical computation the no-slip condition at $t = 0$ can be dropped (Gresho 1991). However, this in no way means that the vortex sheet is not a product of the no-slip condition!

3. Boundary coupling relations

The physical discussions of §2 showed that the ultimate measure of the vorticity creation process at a wall is neither the boundary vorticity $\boldsymbol{\omega}_B$ nor the initial vortex sheet γ_0. Instead, the measure is the boundary vorticity flux or source strength $\boldsymbol{\sigma}$. The same observation applies to the boundary compressing flux $\delta = -\partial\Pi/\partial n$ (e.g. piston motion in a pipe). Consequently, knowledge of $\boldsymbol{\sigma}, \delta$ on a wall also represents the physically most natural boundary conditions in the ω–Π formulation (Wu *et al.* 1992). It is therefore highly desirable to obtain the $\boldsymbol{\sigma}, \delta$ formulae of a Newtonian fluid in their full generality. These formulae are called the boundary ω–Π coupling relations, since they clearly convey the crucial dependence of $\boldsymbol{\sigma}$ on Π_B and of δ on $\boldsymbol{\omega}_B$ due to the adherence condition. Hence, the formulae show the mutual dependence of $\boldsymbol{\omega}$ and Π inside a flow field.

As in the case of unidirectional flow, the desired $\boldsymbol{\sigma}, \delta$ formulae result from taking the tangential and normal components of the Navier–Stokes equation on a wall. A key requirement is that $\boldsymbol{\sigma}$ and δ must be expressed exclusively in terms of quantities at the wall and their tangential derivatives. Such formulae for Stokes–Helmholtz scalar and vector potentials were derived by Wu & Wu (1992). Since the Navier–Stokes equation (1.6) is written in the form of a Stokes–Helmholtz decomposition, it is straightforward to write down the general boundary ω–Π coupling relations for three-dimensional compressible viscous flow over an arbitrarily curved surface, under the adherence condition.

The velocity adherence condition reads

$$\boldsymbol{u} = \boldsymbol{b} \quad \text{at} \quad \partial B, \tag{3.1}$$

where

$$\boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{b}_0(t) + \boldsymbol{W}(t) \times [\boldsymbol{x} - \boldsymbol{x}_0(t)] \tag{3.2}$$

is the velocity of the solid, with $\boldsymbol{x}_0$ being the instantaneous rotating centre and $\boldsymbol{b}_0 = \mathrm{d}\boldsymbol{x}_0/\mathrm{d}t$. Then, the acceleration $\boldsymbol{a}_B$ of a fluid particle sticking to a point on ∂B must equal that of the solid surface at the same point, denoted by $\boldsymbol{a}_b$. Therefore, if a point on ∂B has the Eulerian coordinate $\boldsymbol{x}_B$ at time t, then not only the fluid velocity, but also its acceleration, will be the same as that of a solid surface at $(\boldsymbol{x}_B, t)$:

$$\boldsymbol{a}_B(\boldsymbol{x}_B, t) = \boldsymbol{a}_b(\boldsymbol{x}_B, t). \tag{3.3}$$

The velocity adherence implies acceleration adherence.

In what follows the external body force $\boldsymbol{f}$ in (1.6) will be ignored, because it behaves in exactly the same way as the inertial force $-\boldsymbol{a}$ and can be recovered when desired. By comparing (1.6) with the coupling relations given by Wu & Wu (1992) for general Stokes–Helmholtz potentials, and by using (2.4) and (3.3) (both of which are the consequence of (3.1)), it easily follows that, at each smooth part of ∂B, the boundary compressing flux δ and the boundary vorticity flux $\boldsymbol{\sigma}$ are

$$\delta = -\frac{\partial \Pi}{\partial n} = \boldsymbol{n} \cdot \rho \boldsymbol{a}_b + (\boldsymbol{n} \times \nabla) \cdot (\mu \boldsymbol{\omega}) - 2[(\boldsymbol{\omega}' + \boldsymbol{W}) \times n] \cdot \nabla \mu, \tag{3.4a}$$

$$\boldsymbol{\sigma} = \frac{\partial(\mu \boldsymbol{\omega})}{\partial n} = \boldsymbol{n} \times \rho \boldsymbol{a}_b + (\boldsymbol{n} \times \nabla) \cdot (\Pi \boldsymbol{I} + \mu \boldsymbol{\omega}' \times \boldsymbol{nn}) + 2\vartheta \boldsymbol{n} \times \nabla \mu + \boldsymbol{n}\boldsymbol{\omega}' \cdot \nabla \mu + 2[\boldsymbol{W} \times (\boldsymbol{n} \times \nabla \mu) + \boldsymbol{W}\boldsymbol{n} \cdot \nabla \mu]. \tag{3.4b}$$

For this derivation, operations related to tensor $\boldsymbol{B}$ can be found in Appendix A.

For two-dimensional incompressible flow with $\rho = 1$, (3.4a, b) reduce to the familiar formulae:

$$-\frac{\partial p}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{a}_b + (\boldsymbol{n} \times \nabla) \cdot (\nu \boldsymbol{\omega}), \quad \nu \frac{\partial \boldsymbol{\omega}}{\partial n} = \boldsymbol{n} \times \boldsymbol{a}_b + \boldsymbol{n} \times \nabla p. \tag{3.5a, b}$$

For various computational needs, boundary relations (3.5a, b), particularly (3.5b), have been repeatedly rederived, widely cited, or partly utilized by many authors. Also, the physical implications of (3.5b) have been discussed by Panton (1984), Morton (1984), Reynolds & Carr (1985), and Hunt (1987). The three-dimensional incompressible version of (3.4a, b) was first obtained by Wu (1986) and the extension to compressible flows was briefly given in Wu, Wu & Wu (1987). An entirely different derivation of the vorticity flux formula, starting from the vorticity equation rather than momentum equation (but independent of the constitutive structure of the continuous medium) was presented by Hornung (1988). Fric & Roshko (1989) and Shi, Wu & Wu (1990) used (3.4*b*) to explain the source of all vorticity in the near field of a transverse jet.

Equations (3.4*a*, *b*) show that the compressing flux δ and the vorticity flux $\boldsymbol{\sigma}$ have three sources. Therefore

$$\delta = \delta_a + \delta_\tau + \delta_\mu \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_t + \boldsymbol{\sigma}_\mu, \tag{3.6a, b}$$

where

$$\delta_a = \boldsymbol{n} \cdot \rho \boldsymbol{a}_b \quad \text{and} \quad \boldsymbol{\sigma}_a = \boldsymbol{n} \times \rho \boldsymbol{a}_b \tag{3.7a, b}$$

are the contribution of inertial force;

$$\delta_\tau = (\boldsymbol{n} \times \nabla) \cdot (\mu \boldsymbol{\omega}_B) \quad \text{and} \quad \boldsymbol{\sigma}_t = (\boldsymbol{n} \times \nabla) \cdot (\Pi \boldsymbol{I} + \mu \boldsymbol{\omega}'_B \times \boldsymbol{nn}) \tag{3.8a, b}$$

are contributions of non-uniform distribution of shear and/or normal stress at ∂B, which reflect the coupling between $\boldsymbol{\omega}$ and Π. Finally,

$$\delta_\mu = -2[(\boldsymbol{\omega}' + \boldsymbol{W}) \times \boldsymbol{n})] \cdot \nabla \mu = 2\boldsymbol{\tau} \cdot \nabla \log \mu - 2\boldsymbol{W} \cdot (\boldsymbol{n} \times \nabla \mu), \tag{3.9a}$$

$$\boldsymbol{\sigma}_\mu = 2\vartheta \boldsymbol{n} \times \nabla \mu + \boldsymbol{n}\boldsymbol{\omega}' \cdot \nabla \mu + 2[\boldsymbol{W} \times (\boldsymbol{n} \times \nabla \mu) + \boldsymbol{W}\boldsymbol{n} \cdot \nabla \mu] \tag{3.9b}$$

are the effect of variable viscosity, which, in turn, is the result of the temperature gradient.

For later analysis, it is necessary to further split $\boldsymbol{\sigma}_t$ into two parts:

$$\boldsymbol{\sigma}_t = \boldsymbol{\sigma}_\Pi + \boldsymbol{\sigma}_\tau, \tag{3.10a}$$

where

$$\boldsymbol{\sigma}_\Pi = \boldsymbol{n} \times \nabla \Pi_B \tag{3.10b}$$

is from the normal stress, and

$$\boldsymbol{\sigma}_\tau = (\boldsymbol{n} \times \nabla) \cdot (\mu \boldsymbol{\omega}'_B \times \boldsymbol{nn}) = -(\boldsymbol{n} \times \nabla) \cdot (\boldsymbol{\tau n}) \tag{3.10c}$$

is a novel term from the shear stress.

In summary, among four boundary dynamic quantities ($\mu \boldsymbol{\omega}_B$, Π_B, $\boldsymbol{\sigma}$ and δ) exist two coupling relations (3.4*a*, *b*). These relations result from the orthogonal decomposition of the Navier–Stokes equation at the solid surface and the consequences of velocity adherence, (2.4) and (3.3). Owing to these relations, in the boundary dynamic information set ($\mu \boldsymbol{\omega}_B, \Pi_B, \boldsymbol{\sigma}, \delta$) the number of independent components is reduced to three, say $\mu \boldsymbol{\omega}_B$ and Π_B. This fact is of great importance in the $\boldsymbol{\omega}$–Π formulation (see Wu *et al.* 1992).

It is remarkable that a symmetric and complementary relationship exists between $\boldsymbol{n} \times \mu \boldsymbol{\omega}'_B$ and $\Pi_B \boldsymbol{n}$, and between $\boldsymbol{\sigma}$ and δ. The relationship results from the decomposition of the boundary stress and the 'stress flux' respectively. In short, these two pairs of equations describe the splitting of shearing and compressing processes in their various aspects. It is even more significant to note the close coupling of these two processes, which appears in particular in the expressions of $\boldsymbol{\sigma}$ and δ. This inherent coupling indicates that once a solid surface appears in a flow, any theory that treats $\boldsymbol{\omega}$ and Π separately will be physically incomplete. This fact applies equally to incompressible flows, where there is $\boldsymbol{\omega}$–p coupling.

The $\boldsymbol{\omega}$–Π coupling is a kind of local mechanism. By a corollary of the Stokes theorem

$$\oint_S (\boldsymbol{n} \times \nabla) \circ \mathscr{F} \, dS = 0, \tag{3.11}$$

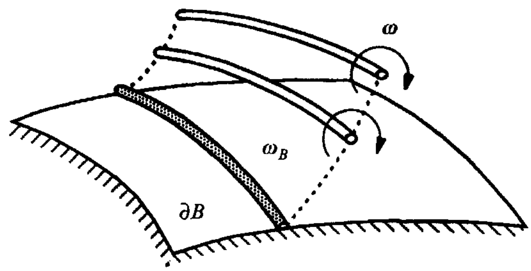

FIGURE 3. The ascending of vortex lines close to the solid surface due to diffusion. In this figure and figure 4 the grey vortex lines represent boundary vorticity and stick to the solid surface.

where $\mathscr{F}$ is any differentiable tensor and $\circ$ denotes any meaningful operation of $\boldsymbol{n}\times\nabla$ and $\mathscr{F}$, it follows from (3.8 *a*, *b*) that

$$\oint_{\partial B} \delta_\tau \, \mathrm{d}S = \oint_{\partial B} \sigma_\Pi \, \mathrm{d}S = \oint_{\partial B} \sigma_\tau \, \mathrm{d}S = 0. \tag{3.12}$$

Clearly, these equations represent some conservation laws which will appear as compatibility conditions when the resultant force formulae are considered (see §5).

It should be noted that for δ_μ and $\boldsymbol{\sigma}_\mu$ a conservation law like (3.12) does not exist. But, for the variable-viscosity term in (1.6) (i.e. $2\boldsymbol{B}\cdot\nabla\mu$) an integral relation can be found by using relations as given in Appendix A:

$$\int_V \boldsymbol{B}\cdot\nabla\mu \, \mathrm{d}V = \boldsymbol{W}\times\oint_{\partial B} \mu \, \mathrm{d}\boldsymbol{S}.$$

Thus, if ∂B is isothermal, the integrated effect of $2\boldsymbol{B}\cdot\nabla\mu$ disappears. Since the effect of variable viscosity is relatively weak, there is no need to return to δ_μ and $\boldsymbol{\sigma}_\mu$.

It is important to see the dual role of $\boldsymbol{\sigma}$ and δ. As the source strengths of $\boldsymbol{\omega}$ and Π, respectively, they are responsible for the creation of $\boldsymbol{\omega}$ and Π directly on the wall. But as fluxes they are also responsible for sending $\boldsymbol{\omega}$ and Π into the flow field. This second role enables clarification of the physical mechanisms by which a $\boldsymbol{\omega}$–Π field goes into the interior of the fluid.

For two- and three-dimensional flows, the role of $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_\Pi$ in sending vorticity to the fluid is essentially the same as in a unidirectional flow case. Owing to diffusion (not convection), vortex lines infinitesimally close to the surface (i.e. inside the hemispheres; but not boundary vortex lines $\boldsymbol{\omega}_B$, which cannot move) spread from, and parallel to, the body surface ∂B (but not necessarily parallel to $\boldsymbol{\omega}_B$ at the foot of $\boldsymbol{n}$). For convenience here, it is called the 'ascending mechanism' (see figure 3). For two-dimensional flow, the only way in which vortex lines can spread away from ∂B is by this mechanism. For three-dimensional attached flow, ascending is still the main part of the whole vorticity flux $\boldsymbol{\sigma}$.

As a part of the ascending mechanism, compressibility directly affects the boundary vorticity flux $\boldsymbol{\sigma}$ through $\boldsymbol{\sigma}_\Pi = \boldsymbol{n}\times\nabla[p-(\lambda+2\mu)\vartheta]$. Denote by $(\;)_\pi$ the component of a vector tangent to the wall; in most cases $|\nabla_\pi p| \gg |\nabla_\pi[(\lambda+2\mu)\vartheta]|$. Moreover, if the flow is steady and ∂B is stationary, the continuity equation gives $\vartheta_B = 0$. Thus, in general, $\boldsymbol{\sigma}$ due to ϑ is weak. However, some important exceptions should not be ignored. One such instance is the formation of a starting vortex and the establishment of the Kutta condition immediately after the impulsive start of an airfoil. Near the trailing edge there must be local regions where the fluid experiences an extremely strong expansion or compression.

The mechanism behind the flux $\boldsymbol{\sigma}_\tau$ due to shear stress is different and exists only for three-dimensional flows. Clearly, it represents $\boldsymbol{\omega}$–$\boldsymbol{\omega}$ coupling rather than $\boldsymbol{\omega}$–Π coupling. This is explained in the next section.

4. Three-dimensional effect on vorticity creation

As mentioned at the beginning of §3, the basic procedure in deriving (3.4*a, b*) is to separate the normal and tangential components of (1.6) applied on the wall by taking scalar and vector products of it with $\boldsymbol{n}$, respectively. Assume $\boldsymbol{f} = 0$ and μ = constant for simplicity. The vector product of (1.6) and $\boldsymbol{n}$ yields

$$\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_\Pi = -\boldsymbol{n} \times (\nabla \times \mu\boldsymbol{\omega}) = \boldsymbol{\sigma} - \nabla(\mu\boldsymbol{\omega}) \cdot \boldsymbol{n} \tag{4.1}$$

on the wall. Hence, comparing (4.1) with (3.4*b*), one sees that the boundary vorticity flux $\boldsymbol{\sigma}_\tau$ due to shearing results from re-expressing $\nabla(\mu\boldsymbol{\omega}) \cdot \boldsymbol{n}$ in terms of $\boldsymbol{\omega}_B$ and its tangential gradient (i.e. (3.10*c*)).

In §2 it was asserted that, at least for unidirectional flow, vorticity creation is entirely caused by the tangential force balance over those fluid hemispheres centred at and sticking to the wall. A problem immediately arises. Equation (4.1) indicates that the tangential viscous force for balancing the body force and the pressure force is

$$-\boldsymbol{n} \times (\nabla \times \mu\boldsymbol{\omega}) = \boldsymbol{\sigma} - \boldsymbol{\sigma}_\tau \tag{4.2}$$

instead of the whole $\boldsymbol{\sigma}$. Thus, $\boldsymbol{\sigma}_\tau$ seems not to appear in this balance nor in any normal balance. This observation led Lyman to propose taking (4.2) as an alternative measure of the vorticity creation strength (see Trefethen & Panton 1990). This measure differs from $\boldsymbol{\sigma}$ only in three-dimensional flows. Once in three dimensions it should first be judged which measure, $\boldsymbol{\sigma}$ or $-\boldsymbol{n} \times (\nabla \times \mu\boldsymbol{\omega})$, represents the actual rate of vorticity creation. The key for making this judgement is to recall that the Navier–Stokes equation represents the force balance per unit volume. Hence for vanishing $\boldsymbol{f}$ and constant μ, the prototype of (4.1) is

$$\boldsymbol{n}_B \times \int_V (\boldsymbol{a} + \nabla\Pi)\,\mathrm{d}V = -\boldsymbol{n}_B \times \oint_{\partial V} \boldsymbol{n} \times \mu\boldsymbol{\omega}\,\mathrm{d}S,$$

where V and ∂V are now the volume and boundary surface of one of those hemispheres. Therefore, $-\boldsymbol{n} \times (\nabla \times \mu\boldsymbol{\omega})$ comes from the tangent component of the integrated shear stress over closed ∂V, of which only a part (the bottom of the hemisphere) is on ∂B.

By contrast, the vorticity creation is a process occurring on any open piece of ∂B. Owing to the partial cancellation of viscous stress $\boldsymbol{n} \times \mu\boldsymbol{\omega}$ over closed ∂V, the tangential component of the volumetric viscous force cannot always represent the creation process completely. It happens in two dimensions, but $\boldsymbol{\sigma}_\tau$ is missing in three dimensions. For the latter case the flux $\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}_\tau$ as part of it, is the only correct measure. It is defined on an open surface, entering as a source term into the integrated vorticity equation over a control volume with a piece of wall as part of its boundary.†

The necessity of having $\boldsymbol{\sigma}_\tau$ as part of the vorticity creation strength is further justified in the reaction phase (§6).

Two subprocesses can be identified in $\boldsymbol{\sigma}_\tau$. By (3.10*c*), there is

$$\boldsymbol{\sigma}_\tau = -\mu\boldsymbol{\omega}' \cdot \nabla\boldsymbol{n} - \boldsymbol{n}\nabla_\pi \cdot (\mu\boldsymbol{\omega}'), \tag{4.3a}$$

or

$$\boldsymbol{\sigma}_\tau = (\boldsymbol{n} \times \boldsymbol{\tau}) \cdot \nabla\boldsymbol{n} - \boldsymbol{n}[\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau})]. \tag{4.3b}$$

† Owing to (3.12), the result will be trivial if the control volume encloses the whole body.

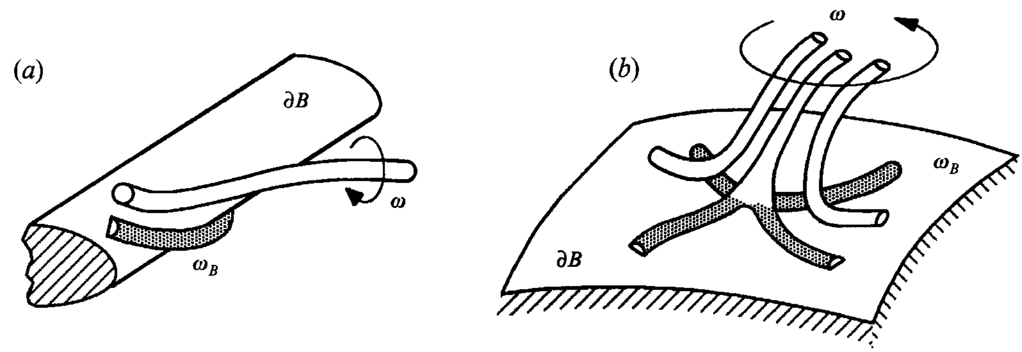

FIGURE 4. Boundary vorticity flux due to ω–ω coupling. (*a*) Ascending of vortex lines close to the wall caused by the wall curvature. (*b*) Upturning of vortex lines due to the curl of shear stress.

In obtaining (4.3*b*), a vector identity given by Wu & Wu (1992), as well as (2.6), is used. It is evident that one subprocess is due to the curvature of ∂B (figure 4*a*), which gives rise to the tangential component of $\boldsymbol{\sigma}_\tau$, denoted by $\boldsymbol{\sigma}_{\tau\pi}$:

$$\boldsymbol{\sigma}_{\tau\pi} = -\mu\boldsymbol{\omega}' \cdot \nabla \boldsymbol{n} = (\boldsymbol{n} \times \boldsymbol{\tau}) \cdot \nabla \boldsymbol{n}. \qquad (4.4a)$$

Again, this is an ascending mechanism. The other subprocess is mainly related to spiral flows above ∂B, which result in the curl of $\boldsymbol{\tau}$ or the two-dimensional divergence of $\boldsymbol{\omega}'_B$ (figure 4*b*), yielding a normal component of $\boldsymbol{\sigma}_\tau$:

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_\tau = \sigma_{\tau 3} = -\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau}) = -\nabla_\pi \cdot (\mu\boldsymbol{\omega}'). \qquad (4.4b)$$

A non-zero $\sigma_{\tau 3}$ implies that vortex lines close to, and originally parallel to, the solid surface have a tendency to turn into the normal direction. Thus this is called the 'upturning mechanism'. The form of (4.4*b*) indicates that this mechanism has its root at the solenoidal property of a vorticity field.

There is a basic difference between ascending and upturning mechanisms. By (2.1*a*), for constant μ the boundary enstrophy flux, say η, is given by

$$\eta \equiv \mu \boldsymbol{n} \cdot \nabla (\tfrac{1}{2}\omega^2) = \boldsymbol{\omega}_B \cdot \boldsymbol{\sigma}.$$

Therefore, on a non-rotating wall with $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ the upturning does not cause an enstrophy change, while the ascending does, including $\boldsymbol{\sigma}_a$, $\boldsymbol{\sigma}_\Pi$ and $\boldsymbol{\sigma}_{\tau\pi}$. For example, in this case there is

$$\eta_\tau = \boldsymbol{\omega}_B \cdot \boldsymbol{\sigma}_\tau = -\mu \boldsymbol{\omega}_B \cdot \nabla \boldsymbol{n} \cdot \boldsymbol{\omega}_B,$$

of which the sign (enstrophy source or sink) only depends on the curvature of ∂B along the direction of boundary vorticity.

Lighthill (1963) pointed out that the normal component of vorticity flux is relatively small, which is generally true for attached flow. When the separation lines (which must exist somewhere on a closed body surface) are approached, however, more careful analysis is required. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be a set of orthogonal unit base vectors movable on ∂B, such that $\boldsymbol{e}_3 = \hat{\boldsymbol{n}}$ and $\boldsymbol{e}_1, \boldsymbol{e}_2$ are along the positive directions of $\boldsymbol{\tau}$ and $\boldsymbol{\omega}$ respectively. Then, for incompressible, three-dimensional steady flow over a stationary curved surface it can be shown that

$$\sigma_{\tau 3} = \left(\frac{\tau_{,2}}{h_2} + \frac{h_{1,2}}{h_1 h_2}\tau\right), \qquad (4.5)$$

where $\tau = \tau_1 = |\boldsymbol{\tau}|$, h_1, h_2 are the Lamé coefficients. Right on the separation line C, there is $\tau_{,2} = 0$. Because $\boldsymbol{\tau}$-lines strongly converge to C (Lighthill 1963), it is expected that

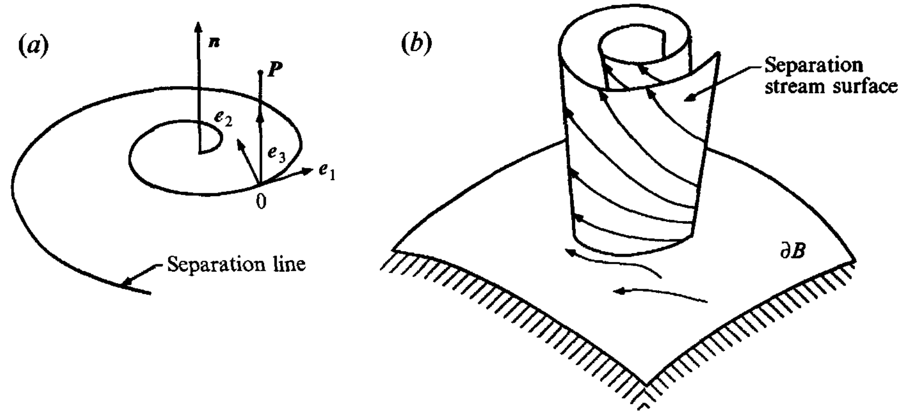

FIGURE 5. Horn vortex formation: (*a*) coordinates, (*b*) rolling-up of separation stream surface.

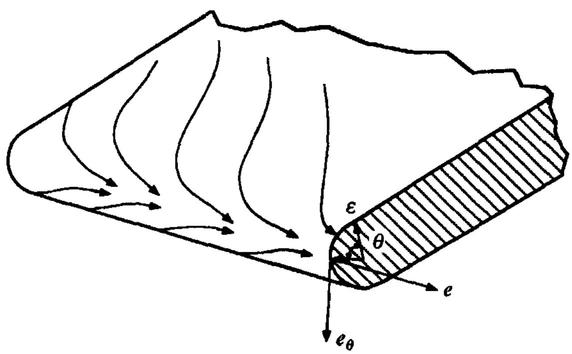

FIGURE 6. Vorticity production at a side edge.

$h_{1,2} = O(1)$ on a narrow strip centred at C, with $\sigma_{\tau 3} = O(\tau)$. Moreover, near a focus of the $\boldsymbol{\tau}$-field, the separation line C rolls up, resulting in many spirally wound strips with $\sigma_{\tau 3} \gg O(\tau)$ (but, by (4.5), $\sigma_{\tau 3} = 0$ right at the focus as seen in figure 5). As the Reynolds number Re approaches infinity, $\tau \sim O(Re^{-\frac{1}{2}})$, and the width of the strip with $h_{1,2} \gg 1$ is $O(Re^{-\frac{1}{2}})$, as is the boundary-layer thickness. On the other hand, a higher Re implies more tightly spiralled separation lines before terminating at the focus. This, in turn, allows for a longer strip and a higher $h_{1,2}$. Thus, it is possible to have a local area with an impulsive $\sigma_{\tau 3}$ even though Re is very high. In fact, this mechanism is responsible for the formation of tornado-like vortices. For more discussion see Wu, Gu & Wu (1988).

Now consider the tangential component of vorticity flux due to shear, $\boldsymbol{\sigma}_{\tau\pi}$. Using the base vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ again, but with $\boldsymbol{e}_1, \boldsymbol{e}_2$ along the two orthogonal principal directions of ∂B, it can be shown that

$$\boldsymbol{\sigma}_{\tau\pi} = -h_1 \kappa_1 \tau_2 \boldsymbol{e}_1 + h_2 \kappa_2 \tau_1 \boldsymbol{e}_2, \tag{4.6}$$

where κ_1, κ_2 are the principal curvatures. If $|\kappa_1| \gg |\kappa_2|$, the main contribution to $\boldsymbol{\sigma}_{\tau\pi}$ is due to the component of $\boldsymbol{\tau}$ in the $\boldsymbol{e}_2$ direction. For example, figure 6 shows a part of the side edge of a plate, which can be idealized as a half-cylinder with radius $\epsilon \rightarrow 0$. Let $(\boldsymbol{e}, \boldsymbol{e}_\theta)$ stand for the unit vectors along the edge and the tangent to the cross-section, respectively. Then, one gets

$$\boldsymbol{\sigma}_{\tau\pi} = \frac{\tau_e}{\epsilon} \boldsymbol{e}_\theta, \tag{4.7}$$

where $\tau_e = \boldsymbol{\tau} \cdot \boldsymbol{e}$. Since ϵ can be made arbitrarily small at the edge, $\boldsymbol{\sigma}_{\tau\pi}$ dominates the whole process by which the vorticity runs into the fluid from any edge where $\tau_e \neq 0$. Wu *et al.* (1988) used this mechanism to explain the wake vortex formation behind a three-dimensional trailing edge. Shi *et al.* (1990) identified a vortex layer near the orifice of a jet in a crossflow formed due to this mechanism.

The above two typical mechanisms for the initial formation of free vortex layers (due to σ_{τ_3} and $\boldsymbol{\sigma}_{\tau\pi}$) occur not only in 'macroscopic' separations, but also in 'microscopic' ones (i.e. local separations inside a boundary layer). Thus, these mechanisms are important in the stability and receptivity of boundary layers. σ_{τ_3} is relevant to the hairpin structure above a flat plate, and $\boldsymbol{\sigma}_{\tau\pi}$ to the Görtler vortex structure along a concave wall.

Since $\boldsymbol{\sigma}_\tau$ is not balanced by an volumetric force over those hemispheres, equations like (4.3)–(4.7) all represent surface force balances along a tangential or the normal direction. These balances are one order higher than (2.5) and do not explicitly enter into the Navier–Stokes equation. But, they can be clearly illustrated by high-order normal and tangential Taylor expansions of a three-dimensional viscous flow field in the neighbourhood of a boundary point, such as those derived by Wu *et al.* (1988) for incompressible steady flow over curved surfaces as the basis of their analysis. The involvement of higher-order force balances in $\boldsymbol{\sigma}$ indicates again that the concept of vorticity creation from a wall is related to the flow behaviour in the neighbourhood of a point rather than on a point alone. In fact, by comparing the expressions

$$\boldsymbol{\tau} = -\mu \boldsymbol{n}\cdot\nabla\boldsymbol{u} \quad \text{and} \quad \boldsymbol{\sigma} = \mu\boldsymbol{n}\cdot\nabla(\nabla\times\boldsymbol{u})$$

one may clearly recognize that, while for the former only the velocity at normally neighbouring points is involved, for the latter the skin-friction at tangentially neighbouring points must be involved as well.

One more remark on the three-dimensional vorticity creation from a wall is in order here. It was said in §2 that even $\boldsymbol{\omega}_B$ is created by $\boldsymbol{\sigma}$; but in three dimensions $\boldsymbol{\sigma}$ contains $\boldsymbol{\omega}_B$. The fact that this situation does not imply a 'chicken–egg' paradox is made clear by the following argument. Assume that G is the fundamental solution of the heat equation in free space, and $\boldsymbol{\omega} = 0$ in the interior of the flow at the impulsive start moment $t = 0$. The effect of convection can be put aside in the present discussion. Then for incompressible flow with $\rho = 1$, it can be shown (Wu *et al.* 1992) that the three-dimensional counterpart of (2.8) reads

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = 2\int_0^t \mathrm{d}\tau \oint_{\partial B} G\boldsymbol{\sigma}\,\mathrm{d}S + \text{convective effect}, \tag{4.8}$$

where $\boldsymbol{\sigma}$ contains $\boldsymbol{\omega}_B$. A singularity of $\boldsymbol{\sigma}$ due to an impulsive start can be handled like (2.11). The physical evolution is: (i) at $t = 0$ the flow is potential except for the singular vortex sheet on the wall, which is already the result of (the singular) $\boldsymbol{\sigma}$ and can in no way yield a $\boldsymbol{\sigma}_\tau$; (ii) for $t \geqslant 0^+$, a finite $\boldsymbol{\sigma}$-distribution continuously adds new $\boldsymbol{\omega}_B$ to the left by diffusing the initial vortex sheet (see (2.12)); and (iii) then, the $\boldsymbol{\omega}_B$ distribution gradually evolves to form a $\boldsymbol{\sigma}_\tau$. Therefore, $\boldsymbol{\sigma}_\tau$ is ultimately an accumulated result of $\boldsymbol{\sigma}_\Pi$ and $\boldsymbol{\sigma}_a$.

Finally, the boundary compressing flux δ can be discussed in a similar but much simpler way. Equation (3.4*a*) represents the dynamic balance along the normal direction, and among the inertial force, pressure gradient and non-uniform shearing. In particular, if the wall acceleration $\boldsymbol{a}_b = 0$ and μ is constant, then

$$\delta = \delta_\tau, \quad \text{i.e.} \quad \boldsymbol{n}\cdot\nabla\Pi = \nabla_\pi\cdot\boldsymbol{\tau} \quad \text{on} \quad \partial B, \tag{4.9}$$

showing again the close coupling of shearing and compressing processes.

The physical meaning of compressing flux δ_τ due to shear can be clearly seen from (4.9). A non-uniform distribution of shear stress $\boldsymbol{\tau}$, particularly its two-dimensional 'source' or 'sink', produces a compressing flux or normal pressure force. This is precisely in contrast to the mechanisms of $\boldsymbol{\sigma}_\Pi$, the vorticity flux due to normal stress. Note that δ_τ is also highly localized and is important only near flow-separation regions. In fact, for incompressible flow there is

$$\delta_\tau = -\mu \boldsymbol{n} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}_\pi \cdot \boldsymbol{u}_\pi) = \mu(\boldsymbol{n} \cdot \boldsymbol{\nabla})^2 u_3, \tag{4.10}$$

representing the upturn (or downturn) of streamlines, which relates to flow separation or attachment (Wu *et al.* 1988).

Part 2. Force and moment reaction to the body by the created ω–Π field

The roles of shearing and compressing in the local reaction are clearly reflected by (2.5). It is the boundary ω–Π that exclusively determines the shear and normal stresses. The situation is different, however, for integrated force and moment. Although the total force is directly obtainable by integrating (2.5) over the body surface, part of the stresses are cancelled out in the integration process. The static pressure on a body surface immersed in a still fluid provides a trivial example. In fact, one of the main results of this study on the reaction phase is a systematic proof of a fundamental fact that, instead of the boundary values of ω–Π, their net fluxes contribute to the total force and moment.

5. General discussion

For incompressible flows, it is well known that the integrated linear and angular momentum of a fluid body can be expressed in terms of the first and second vectorial moments of vorticity. Thus, the rate of change of these integrals directly leads to formulae for force $\boldsymbol{F}$ and moment $\boldsymbol{L}$ in terms of vorticity (e.g. Wu 1981). For example, let V be the material volume of the fluid, d the spatial dimensionality and $k = d-1$, then

$$\boldsymbol{F} = -\frac{\rho}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{x} \times \boldsymbol{\omega}\,\mathrm{d}V + \frac{\rho}{k}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial B} \boldsymbol{x} \times (\boldsymbol{n} \times \boldsymbol{b})\,\mathrm{d}S. \tag{5.1}$$

Lighthill (1986*b*) reported that this result is widely used in estimating the vortex-flow force on an offshore structure. The physical implication of (5.1) is also clear (Batchelor 1967). For a far-field observer, the rate of change of the integrated vorticity moment can be viewed as the rate of increase of the vectorial area spanned by a closed vortex tube. This consists of the 'bound vortex' associated with a solid body (i.e. the net effect of the boundary layer surrounding the body), the continuously elongated wake vortices and the starting vortices (see figure 7).

The same physics can be made clearer as an observer moves close to the above vortex system. For example, the observer can move to a control surface downstream of the body but far upstream of the starting vortices (a 'wake plane'). A general wake-plane analysis for compressible steady flow was given by Wu & Wu (1989). The elongation rate of the wake vortex tubes in figure 7 appears as the convecting flux of the integrated vorticity moment across the wake plane. Although this wake-plane analysis is of relevance to aerodynamics, a more interesting view is obtained if the observer moves even closer, tracing the source of vortex system, i.e. the body surface ∂B. Then it is

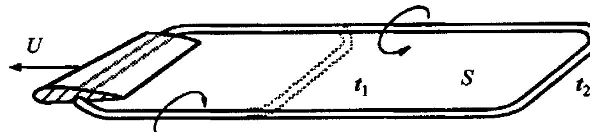

FIGURE 7. The increase of area S spanned by a closed-line vortex.

evident that the increase of S (figure 7), or the steady convecting flux of vorticity moment across the wake plane, must come from the continuous vorticity production from ∂B. This, in turn, must relate to the boundary vorticity flux rather than boundary ω–Π itself. Therefore, the $\boldsymbol{F}, \boldsymbol{L}$ formulae, once expressed in terms of integrals over ∂B, should have a boundary vorticity flux explicitly in their integrands. This will be explored in §6, and is the key theoretical result of the reaction phase.

Except for the control-volume-integral type mentioned above, which is convenient for steady flows only, the integrated force $\boldsymbol{F}$ and moment $\boldsymbol{L}$ are usually expressed in two ways, i.e. in terms of integrals over the material volume V of fluid, and that over the body surface. The task is, therefore, to extend previously known results (mainly for incompressible flows) in terms of vorticity to the most general forms of these two types of $\boldsymbol{F}, \boldsymbol{L}$ formulae in the ω–Π interpretation. The emphasis will be on the second type. A systematic way of doing so is to transform the corresponding formulae in the $\boldsymbol{u}$–p interpretation by a series of vectorial identities using the Gauss and Stokes theorems. Then the mathematical structure of the resulting formulae can be seen clearly. No possible valuable expressions of $\boldsymbol{F}$ and $\boldsymbol{L}$ will be missing. These identities are stated in Appendix B. For convenience, the familiar $\boldsymbol{F}, \boldsymbol{L}$ formulae of the two types in the $\boldsymbol{u}$–p interpretation are listed here. Assuming that external body force is neglected, for the first type one has

$$\boldsymbol{F} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \boldsymbol{u}\,\mathrm{d}V = -\int_V \rho \boldsymbol{a}\,\mathrm{d}V \qquad (5.2a, b)$$

and

$$\boldsymbol{L} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{r}\times\rho \boldsymbol{u}\,\mathrm{d}V = -\int_V \boldsymbol{r}\times\rho \boldsymbol{a}\,\mathrm{d}V, \qquad (5.3a, b)$$

where $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}_0$, with $\boldsymbol{x}_0$ as the fixed point around which the moment is taken. Without losing generality, one takes $\boldsymbol{x}_0 = 0$ exclusively. Then by (2.5), the second type of $\boldsymbol{F}, \boldsymbol{L}$ formulae in the $\boldsymbol{u}$–p interpretation are (although in terms of ω, Π already)

$$\boldsymbol{F} = -\oint_{\partial B} \Pi \hat{\boldsymbol{n}}\,\mathrm{d}S + \oint_{\partial B} \mu \boldsymbol{\omega}' \times \hat{\boldsymbol{n}}\,\mathrm{d}S, \qquad (5.4)$$

$$\boldsymbol{L} = -\oint_{\partial B} \boldsymbol{x}\times \Pi \hat{\boldsymbol{n}}\,\mathrm{d}S + \oint_{\partial B} \boldsymbol{x}\times(\mu \boldsymbol{\omega}' \times \hat{\boldsymbol{n}})\,\mathrm{d}S. \qquad (5.5)$$

An apparent difference between the resultant force formulae in the $\boldsymbol{u}$–p and ω–Π interpretations is that, as can be seen from comparing the dimensions of $\boldsymbol{u}$ and $\boldsymbol{\omega}$, and exemplified by contrasting (5.2a) and (5.1), the position vector $\boldsymbol{x}$ always enters in the formulae of the latter. The total force is, of course, independent of the location of the origin of $\boldsymbol{x}$. Thus, a compatibility condition must exist to ensure this fact. To see this in a general way, let $\mathscr{F}$ be any tensor, $\boldsymbol{x}\circ\mathscr{F}$ be any kind of meaningful product of $\boldsymbol{x}$ and $\mathscr{F}$, and I be the integration operator (over surface or volume). Then the above independence implies that

$$\mathrm{I}\{(\boldsymbol{x}+\boldsymbol{x}_0)\circ\mathscr{F}\} = \mathrm{I}\{\boldsymbol{x}\circ\mathscr{F}\},$$

where x_0 is a constant vector. It follows from the arbitrariness of x_0 that

$$\mathrm{I}\{\mathscr{F}\} = 0. \tag{5.6}$$

This indicates that only those tensors whose integrated effect vanishes can be used to construct a single $\mathrm{I}\{x \circ \mathscr{F}\}$-type formula independent of the origin. One may find that the formulae obtained all satisfy the compatibility condition (5.6), either trivially or by representing some conservation laws which can serve as integral error-control conditions for numerical calculation, no matter what formulation is used.

Another difference between the F, L formulae in the u–p and ω–Π interpretations is that in the latter case the formulae do not have a unique form. It turns out that F, L can be expressed in terms of vorticity alone, and sometimes, equivalently in terms of compressing variable alone. Thus, one has a 'shearing interpretation' and a 'compressing interpretation'. But only the former covers the whole variety of F, L formulae. As a scalar process, the compressing interpretation merely yields a couple of results of some interest. Once again, therefore, the emphasis is on the shearing interpretation. The parallel existence of these alternative interpretations in no way implies that only one process, shearing or compressing, dictates the flow. Rather, the possibility of having these alternatives (and hence their infinite combinations) is just more evidence of ω–Π coupling.

6. Force and moment integrals: the 'shearing interpretation'

By comparing (5.2*a*) and (5.3*a*) with identities (B 2) and (B 3) in Appendix B, for incompressible flows the force formula (5.1), as well as a corresponding moment formula, is obtained. The compatibility condition of (5.1) is evidently the Föppl conservation law of total vorticity. However, these results do not easily extend to compressible flows, unless one introduces a physically meaningless vector $\nabla \times (\rho u)$ which is not the angular momentum of a fluid particle or a finite fluid body. In fact, more instructive results in terms of material volume integrals come from (5.2*b*) and (5.3*b*) which, by using (B 2) and (B 3) again, yield

$$F = -\frac{1}{k}\int_V x \times (\nabla \times \rho a)\,\mathrm{d}V + \frac{1}{k}\oint_{\partial V} x \times \sigma_a\,\mathrm{d}S, \tag{6.1}$$

$$L = -\frac{1}{d}\int_V x \times [x \times (\nabla \times \rho a)]\,\mathrm{d}V + \frac{1}{d}\oint_{\partial V} \rho[(x \times n)(x \cdot a) - (x \times a)(x \cdot n)]\,\mathrm{d}S \tag{6.2a}$$

or

$$L = \frac{1}{2}\int_V x^2 \nabla \times \rho a\,\mathrm{d}V - \frac{1}{2}\oint_{\partial V} x^2 \sigma_a\,\mathrm{d}S, \tag{6.2b}$$

and for $d = 3$ only,

$$L = -\int_V x x \cdot (\nabla \times \rho a)\,\mathrm{d}V + \oint_{\partial V} \rho x a \cdot (x \times n)\,\mathrm{d}S. \tag{6.2c}$$

Here $\sigma_a = n \times \rho a$ stands for the tangential inertial force on ∂V. Like (5.1), both (6.1) and (6.2) are free from the constitutive structure of the fluid. The implication becomes more apparent if one substitutes (1.5) with both the constant μ and $f = 0$. In particular, for a stationary body (assume $\partial V = \partial B + \partial V_\infty$)

$$F = -\frac{1}{k}\int_V x \times \nabla^2(\mu\omega)\,\mathrm{d}V + \frac{1}{k}\oint_{\partial V_\infty} x \times \sigma_a\,\mathrm{d}S \tag{6.3}$$

and, say,
$$L = \frac{1}{2}\int_V x^2\nabla^2(\mu\omega)\,dV - \frac{1}{2}\oint_{\partial V_\infty} x^2\sigma_a\,dS. \qquad (6.4)$$

These are indeed formulae in terms of shearing variable $\mu\omega$ only. The compatibility condition of (6.3) is the conservation law (3.12). Note that (6.3) extends the D'Alembert paradox to any compressible rotational flow of an ideal fluid over a stationary body with $|a| = o(x^{-3})$ at infinity. (Exceptions include acoustic radiation and supersonic flow, for which the surface integral over ∂V_∞ is not zero.)

Now, the body surface integrals (5.4) and (5.5) can be recasted to the 'shearing interpretation'. This can be done by transforming (6.3) and (6.4) through integration by parts (Wu 1987). A more direct approach, however, is to utilize the integral identities (B 10) and (B 13). In fact, (B 7) and (B 8) correspond to (5.4) and (5.5) respectively, provided that
$$f = -\rho a, \quad \phi = -\Pi, \quad A = \mu\omega'.$$
It is natural to decompose each of the F, L into contributions due to normal and tangential stresses. Here, they are denoted as F_Π, F_τ and L_Π, L_τ, which are not necessarily orthogonal to each other.

From (3.10b, c), (B 9a, c), (B 10a) and (B 11) it follows that
$$F_\Pi = -\frac{1}{k}\oint_{\partial B} x\times\sigma_\Pi\,dS \quad \text{for both two- and three-dimensional flow} \qquad (6.5a)$$
and
$$F_\tau = -\oint_{\partial B} x\times\sigma_\tau\,dS \quad \text{for three-dimensional flow,} \qquad (6.5b)$$
again with (3.12) as their compatibility conditions. Therefore, as expected, for three-dimensional flow F is expressed in terms of boundary vorticity flux solely:
$$F = -\oint_{\partial B} x\times(\tfrac{1}{2}\sigma_\Pi+\sigma_\tau)\,dS \quad \text{for three-dimensional flow.} \qquad (6.6)$$
This remarkable result was first obtained by Wu (1987) for incompressible flow over a stationary body, and extended to compressible flow by Wu *et al.* (1987) without giving a derivation.

Unfortunately, owing to the different tensorial properties of Π and τ, it is impossible to express F as the integration of a single $x\times\sigma_t$, where $\sigma_t = \sigma_\Pi+\sigma_\tau$ is defined by (3.8b), i.e. the boundary vorticity flux due to total stress $t = -\Pi n+\tau$. Moreover, because of the lack of σ_τ in two-dimensional flow one has to be satisfied with
$$F = -\oint_{\partial B} x\times\sigma_\Pi\,dS + \oint_{\partial B} \mu\omega'\times\hat{n}\,dS. \qquad (6.7)$$
Similarly, from (B 13)
$$L_\Pi = -\frac{1}{d}\oint_{\partial B} x\times(x\times\sigma_\Pi)\,dS = \frac{1}{2}\oint_{\partial B} x^2\sigma_\Pi\,dS \qquad (6.8a, b)$$
for two- and three-dimensional flows, and for three-dimensional flow only,
$$L_\Pi = -\oint_{\partial B} x(x\cdot\sigma_\Pi)\,dS. \qquad (6.8c)$$

Then, by using any two identities of (B 13a–c),

$$L_\tau = \oint_{\partial B} (\tfrac{1}{2}x^2\boldsymbol{\sigma}_\tau - \boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{\sigma}_\tau)\,\mathrm{d}S \tag{6.9}$$

for three-dimensional flow only. Therefore, from (6.8),

$$\boldsymbol{L} = \int_{\partial B} (\tfrac{1}{2}x^2\boldsymbol{\sigma} - \boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{\sigma}_\tau)\,\mathrm{d}S \quad \text{for three-dimensional flow.} \tag{6.10}$$

As a simple application of (6.6), a calculation of the total force of the steady Stokes flow over a sphere of radius a may exhibit some of the features of the shearing interpretation. Take the spherical polar coordinates (r, θ, ϕ) with the origin at the sphere's centre and $\theta = 0$ along the direction of oncoming flow $\boldsymbol{U}$. Then (e.g. Batchelor 1967)

$$\boldsymbol{\omega} = -\frac{3}{2}aU\frac{\sin\theta}{r^2}\boldsymbol{e}_\phi, \quad p = p_\infty - \frac{3}{2}\mu aU\frac{\cos\theta}{r^2}.$$

Now $\boldsymbol{\sigma} = \boldsymbol{e}_\phi\,\sigma = -\mu\boldsymbol{e}_\phi(\partial\omega/\partial r)_a$, and by (3.10$b$, c), vorticity fluxes due to p and $\boldsymbol{\tau}$ are

$$\sigma_p = -\frac{1}{a}\frac{\partial p}{\partial\theta} = -\frac{3}{2}\mu U\frac{\sin\theta}{a^2}, \quad \sigma_\tau = \sigma_{\tau\pi} = \frac{\mu}{a}\omega = \sigma_p. \tag{6.11a, b}$$

Equations (6.6) and (6.11b) show that the pressure distribution provides one-third of the total drag, while the skin friction provides the other two-thirds. Substitution of (6.11) into (6.6) gives the classical result

$$D = 6a\pi\mu U \quad \text{or} \quad C_D = 24/Re.$$

Note that the vorticity flux caused by the tangential pressure gradient is by no means an inviscid contribution. But it is hard to explain in the conventional $\boldsymbol{u}$–p interpretation. For example, the contribution of pressure to the drag has been said to be due to 'the viscous component of the normal stress' (Illingworth 1963). This identification is quite obscure without a detailed stress analysis (Batchelor 1967). However, the relevant viscous mechanism is clearly explained in §§2 and 4. The total force $\boldsymbol{F}$ due to $\boldsymbol{\sigma}_\Pi$ and $\boldsymbol{\sigma}_{\tau\pi}$ is recognized as related to the rate of work required for raising vortex lines near the surface due to the $\boldsymbol{\omega}$–p coupling and $\boldsymbol{\omega}$–$\boldsymbol{\omega}$ coupling respectively, which have their roots in viscosity and the adherence condition.

Although (6.6) and (6.11) are of value in numerical computations (for two-dimensional incompressible flow (6.5a) is well-known to computational fluid dynamicists), their role in understanding the physics of dynamic interactions is more remarkable. In Part 1 the action of a body on a surrounding vorticity field is exclusively represented by the boundary vorticity flux. That same flux now reflects the integrated reaction of the vorticity created on the body. Therefore, the action and reaction phases are conceptually unified in the 'shearing interpretation'.

It is interesting to look at the part of the stresses which has no net contribution to $\boldsymbol{F}$ and $\boldsymbol{L}$. Take the total force as an example. First, from identity (B 14) it is seen that this kind of normal stress has the form $k^{-1}(\boldsymbol{n}\times\nabla)\times(\Pi\boldsymbol{x})$. This is easily understood. For instance, consider a uniform inviscid incompressible flow over a stationary body. This flow is purely kinematic from the viewpoint of vorticity dynamics. Thus, the pressure distribution $p\boldsymbol{n}$ over ∂B can be entirely rewritten as $k^{-1}(\boldsymbol{n}\times\nabla)\times(p\boldsymbol{x})$, such that $\boldsymbol{F} = 0$ for both two-dimensional and three-dimensional flows. This is another proof of the D'Alembert paradox. Therefore, like (6.3), all normal stresses with kinematic and static background are automatically filtered out when one switches from (5.4) to (6.6). What is left are the normal stresses due to viscous dynamic interaction, which must cause a boundary vorticity flux. The $\boldsymbol{\sigma}_p$ in the Stokes flow over a sphere is of this type.

Next, in viscous dynamic stress there is still a component which may cancel during the integration. By identity (B 15), this component in shear stress has the form

$$(\boldsymbol{n}\times\nabla)\cdot[\tau(\boldsymbol{x}\times\boldsymbol{n})] = \boldsymbol{n}\cdot[\nabla\times(\tau\boldsymbol{x}\times\boldsymbol{n})].$$

Then (4.3) shows that the corresponding boundary vorticity must be such that: (i) it is two-dimensionally divergence-free on ∂B, and (ii) along its direction the surface ∂B has no curvature. On a thin wing of finite span there are often some local two-dimensional regions (for instance the mid-span region of the leeside of a delta wing at a moderate angle of attack). One now sees that in the shearing interpretation these regions have no contribution to the total friction force $\boldsymbol{F}_\tau$ (but they do if (5.4) is used).

In §4 it was mentioned that, in comparison with vorticity flux $\boldsymbol{\sigma}_\Pi$ due to compressing, the flux $\boldsymbol{\sigma}_\tau$ due to shearing is much smaller for attached-flow areas on ∂B, particularly when the Reynolds number is large. However, there must be some vortex lines close to the surface breaking away from the surface somewhere that are closely related to the flow separation. Now one sees that this mechanism is finally responsible for the overall friction drag in the shearing interpretation.

7. Force integrals: the 'compressing interpretation'

If one tries to recast various $\boldsymbol{F}, \boldsymbol{L}$ formulae in terms of the scalar compressing process alone, the result is much less fruitful than in the shearing interpretation. First, the corollary III shown in Appendix B excludes any possibility of expressing the moment $\boldsymbol{L}$ solely in terms of some volume integral of the moment of $\nabla\cdot(\rho\boldsymbol{a})$. Second, since there is no compressing flux due to normal stress, it is also impossible to express $\boldsymbol{F}_\Pi$ and $\boldsymbol{L}_\Pi$, defined in §6, in terms of that flux. Third, the corollary VI shown in Appendix B also indicates that it impossible to recast $\boldsymbol{L}_\tau$ (defined in §6) in the 'compressing interpretation'. Therefore, using this interpretation there is no integrated moment formula at all, nor the counterpart of force formulae like (6.5a) and (6.6). Finally, if $\boldsymbol{f} = \rho\boldsymbol{u}$ is set in (B 1) and substituted into (5.2a), a force due to potential flow only is obtained, since the no-slip condition cannot to be imposed. Hence (5.1) has no counterpart in the compressing interpretation either. The only possibility for the 'compressing interpretation', therefore, is to use a counterpart of (6.1) and (6.3), the force integral over material volume, and that of (6.5b), the tangential stress integral over the body surface.

Comparing (5.2b) and (B 1), it follows that

$$\boldsymbol{F} = \int_V \boldsymbol{x}\nabla\cdot(\rho\boldsymbol{a})\,\mathrm{d}V - \oint_{\partial V} \boldsymbol{x}\delta_a\,\mathrm{d}S, \tag{7.1}$$

where $\delta_a = \boldsymbol{n}\cdot\rho\boldsymbol{a}$ stands for the normal inertial force on ∂V. It can be proven that (7.1) is indeed equivalent to (6.1). In particular, for Newtonian fluid flow with the constant μ over a stationary body, one has the counterpart of (6.3):

$$\boldsymbol{F} = -\int_V \boldsymbol{x}\nabla^2\Pi\,\mathrm{d}V - \oint_{\partial V_\infty} \boldsymbol{x}\boldsymbol{n}\cdot\rho\boldsymbol{a}\,\mathrm{d}S. \tag{7.2}$$

Thus, as long as the full adherence condition is imposed, the total force, including overall friction, can be expressed solely in terms of the compressing process.

Equation (7.1) has an interesting consequence. For incompressible flow over a stationary body, by using the familiar formula valid for any continua (Serrin 1959, p. 168)

$$\nabla \cdot \boldsymbol{a} = \frac{\mathrm{D}\vartheta}{\mathrm{D}t} + \boldsymbol{D}:\boldsymbol{D} - \tfrac{1}{2}\omega^2,$$

(7.1) can be rewritten as

$$\boldsymbol{F} = \rho \int_V \boldsymbol{x}(\boldsymbol{D}:\boldsymbol{D} - \tfrac{1}{2}\omega^2)\,\mathrm{d}V. \tag{7.3}$$

Therefore, for an incompressible flow with $\boldsymbol{a}\cdot\boldsymbol{n} = 0$ on its boundary the local difference of $\boldsymbol{D}:\boldsymbol{D}$ and $\frac{1}{2}\omega^2$, after taking a moment, integrate to the local force. This is true even though both $\boldsymbol{D}:\boldsymbol{D}$ and $\frac{1}{2}\omega^2$ integrate to the same total dissipation (Serrin 1959, p. 250).

The total friction force $\boldsymbol{F}_\tau$ remains in the compressing interpretation. Equation (3.8*a*) suggests that the scalar δ_A defined by (B 9*b*) should be identified as $\delta_A = \delta_\tau - (\boldsymbol{n}\times\nabla)\cdot(2\mu\boldsymbol{W})$. Thus from (B 11*b*), for the force due to tangential stress we have

$$\boldsymbol{F}_\tau = \oint_{\partial B} \mu\boldsymbol{\omega}'\times\hat{\boldsymbol{n}}\,\mathrm{d}S = \oint_{\partial B} \boldsymbol{x}\delta_\tau\,\mathrm{d}S - 2\boldsymbol{W}\times\oint_{\partial B}\boldsymbol{n}\mu\,\mathrm{d}S. \tag{7.4}$$

But for the total force $\boldsymbol{F} = \boldsymbol{F}_\Pi + \boldsymbol{F}_\tau$ (if $\boldsymbol{W} = 0$) one has to be satisfied with

$$\boldsymbol{F} = \oint_{\partial B} (-\Pi\hat{\boldsymbol{n}} + \boldsymbol{x}\delta_\tau)\,\mathrm{d}S, \tag{7.5}$$

the counterpart of (6.7). Even though the compressing interpretation for the total force formula is not as simple as (6.6), the overall friction with constant μ provides simple symmetric results:

$$\boldsymbol{F}_\tau = \oint_{\partial B} \boldsymbol{x}\delta_\tau\,\mathrm{d}S \quad \text{for two- and three-dimensional flow} \tag{7.6a}$$

$$= -\oint_{\partial B} \boldsymbol{x}\times\boldsymbol{\sigma}_\tau\,\mathrm{d}S \quad \text{for three-dimensional flow only,} \tag{7.6b}$$

with δ_τ and $\boldsymbol{\sigma}_\tau$ being the two fluxes due to shearing.

Unlike the origin of δ_τ (§4) where the shearing interpretation of $\boldsymbol{F}_\tau$ due to $\sigma_{\tau 3}$ comes from the upturn of the close-surface vortex lines, in the compressing interpretation it comes from the upturn or downturn of the close-surface streamlines (i.e. from the flow separation and attachment). In three-dimensional flow for which both (7.6*a*) and (7.6*b*) hold, the local areas of ∂B where $\boldsymbol{x}\delta_\tau$ is important are, in general, different from those where $\boldsymbol{x}\times\boldsymbol{\sigma}_\tau$ is important. The shearing and compressing interpretations of $\boldsymbol{F}_\tau$ provide an alternative along with their own key areas.

In the new force and moment formulae derived in this section a significant feature should be noted: at high Reynolds numbers there are many situations where the integrands $\boldsymbol{\sigma}_\Pi$ and $\boldsymbol{\sigma}_\tau$ in (6.6) and (6.10), as well as δ_τ in (7.6*a*), are more localized than Π and $\boldsymbol{\tau}$ on the body surface. Indeed, for a Blasius flat plate, $\boldsymbol{\sigma}_\Pi \neq 0$ only at its leading edge. Even if the flow separates from such a plate, $\boldsymbol{\sigma}_\Pi$ has a substantial value only near the separation point. Moreover, §4 shows that $\boldsymbol{\sigma}_\tau$ and δ_τ are negligible except for a local region close to separation and reattachment lines. In fact, the key areas where $\boldsymbol{\sigma}_\tau$ and δ_τ differ from zero appreciably are located precisely around the critical points of the $\boldsymbol{\omega}_B$

field (or the $\boldsymbol{\tau}$-field) on the body surface. This implies that the theory on critical points and their topological structures is intrinsically related to the ω–Π interpretation. While this relationship is itself a topic worth further exploration, here (7.6*a*) is used to make a rough friction-drag estimation for two-dimensional incompressible steady flow merely to illustrate the relevance of critical points to the overall friction $\boldsymbol{F}_\tau$:

$$\boldsymbol{F}_\tau = \oint_{\partial B} \boldsymbol{x}\delta_\tau \,\mathrm{d}S = \oint_{\partial B} \boldsymbol{x}\delta \,\mathrm{d}S = -\oint_{\partial B} \boldsymbol{x}\hat{\boldsymbol{n}}\cdot\nabla p\,\mathrm{d}S \tag{7.7a}$$

$$= \oint_{\partial B} \boldsymbol{x}\nabla_\pi\cdot\boldsymbol{\tau}\,\mathrm{d}S. \tag{7.7b}$$

For simplicity, assume that the Reynolds number is sufficiently high so that only in the region of a separation (or attachment) point $\nabla_\pi\cdot\boldsymbol{\tau} = -\hat{\boldsymbol{n}}\cdot\nabla p \neq 0$. Then let N pairs of attachment and separation points on ∂B (denoted by $a_i, s_i, i = 1, \ldots, N$) separate from each other, such that for each point, say a_i, there is an isolated area S_{a_i} (of unit length along the third dimension), having a non-zero contribution to the integral in (7.7*a, b*). This contribution can be written as $\boldsymbol{x}_{a_i} J_{a_i}$, where $\boldsymbol{x}_{a_i}$ is the mean position vector over S_{a_i}, and

$$J_{a_i} = -\int_{S_{a_i}} \nabla_\pi\cdot\boldsymbol{\tau}\,\mathrm{d}S = \int_{S_{a_i}} \boldsymbol{n}\cdot\nabla p\,\mathrm{d}S < 0$$

is the total 'source' of $\boldsymbol{\tau}$ or 'sink' of $\boldsymbol{n}\cdot\nabla p$ on S_{a_i}. On the other hand, $J_{s_i} > 0$ is the total 'sink' of $\boldsymbol{\tau}$ or total 'source' of $\boldsymbol{v}\cdot\nabla p$ on S_{s_i}. Then (7.7*a*) or (7.7*b*) yields

$$\boldsymbol{F}_\tau \sim \sum_{i=1}^{N} (\boldsymbol{x}_{a_i} J_{a_i} + \boldsymbol{x}_{s_i} J_{s_i}), \tag{7.8}$$

with the compatibility condition

$$\sum_{i=1}^{N} (J_{a_i} + J_{s_i}) \sim 0. \tag{7.9}$$

Figure 8 shows some relevant patterns with a closed wake for which $\boldsymbol{F}_\tau$ can be easily inferred qualitatively by using (7.8). For the flat plate of figure 8(*a*), one usually determines the value of $\boldsymbol{\tau}$ along the whole surface but neglects the effect of leading and trailing edges. Then $\boldsymbol{F}_\tau$ follows from (5.4). In contrast, when using (7.8), only the small regions around a and s are important. These two approaches are equivalent. If (5.4) is used, although skin-friction $\boldsymbol{\tau}$ varies quite sharply at a and s, due to the smallness of S_a and S_s, the edge effect can indeed be omitted. On the other hand, if (7.8) is used, because of the smallness of S_a and S_s, $|\nabla_\pi\cdot\boldsymbol{\tau}|$ or $|\boldsymbol{n}\cdot\nabla p|$ becomes so large at a and s that the moment of J_a and J_s precisely represents the whole integration of $\boldsymbol{\tau}$. As the length of the plate increases, the integration area becomes larger in (5.4), simply corresponding to the larger $\boldsymbol{x}_s$ and $\boldsymbol{x}_a$ in (7.8).

Moreover, if the origin of the coordinates moves to the leading (or trailing) edge, then only the trailing (or leading) edge contributes to $\boldsymbol{F}_\tau$ (associated by a larger x). If the origin is at the leading edge, J_s should precisely reflect the accumulated effect of the whole upstream boundary layer. But, if the origin is at the trailing edge, J_a should precisely predict the whole downstream development of the layer. This can also be seen from (7.9).

In figure 8(*b*), s_1 and a_8 induce a thrust, cancelling some friction drag due to a_1 and s_2. In figure 8(*c*) if $J_{a_1} \approx J_{a_2}$, then $F_{\tau_x} \sim 2x_s J_s$ and $F_{\tau_y} = 0$. Figure 8(*d*) is an example

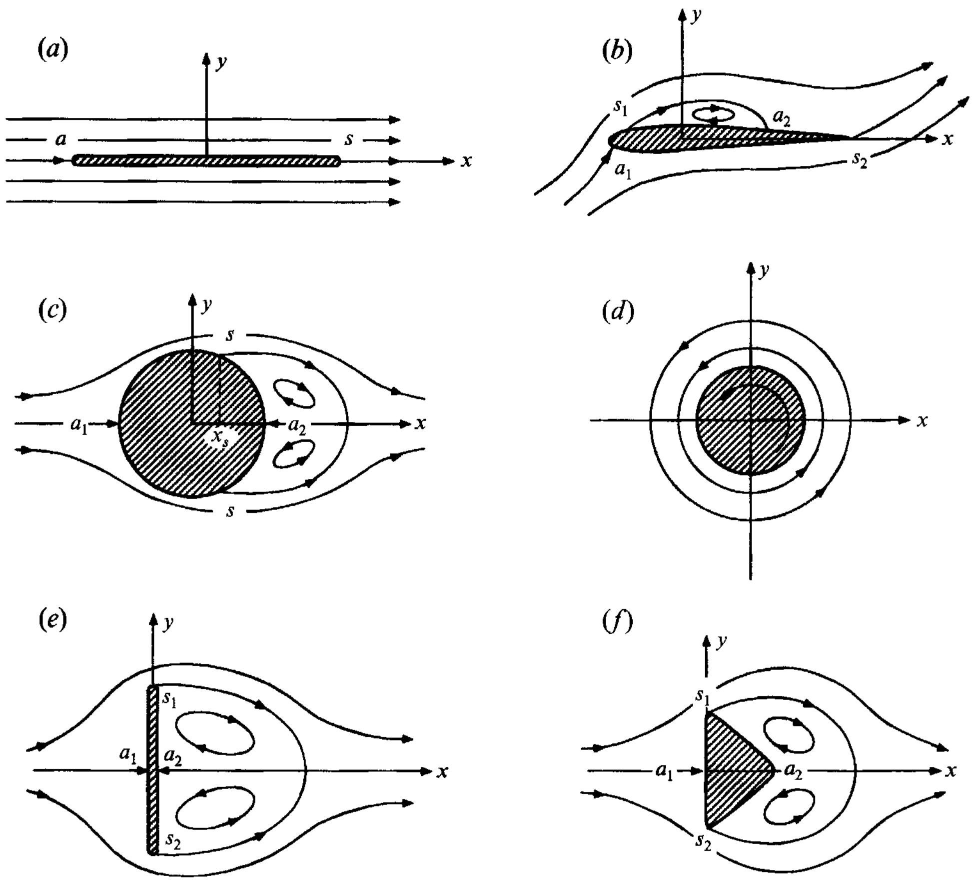

FIGURE 8. Typical two-dimensional steady flow patterns.

without attachment and separation: $F_\tau = 0$. Then, F_τ is still zero in figure 8(*e*). The overall drag comes from F_Π only. Finally, in figure 8(*f*), F_τ is solely due to a_2, resulting in a thrust. A similar discussion on the F_τ source in three-dimensional flows, in terms of σ_τ, was given by Wu (1987).

These examples show that predicting (and possibly controlling) the global behaviour of a flow field by concentrating on local areas where p and τ vary sharply, and meanwhile covering the whole surface by taking a moment, is a significant characteristic of the ω–Π interpretation. Moreover, using the above results needs topological knowledge of the critical points as a prerequisite, while at the same time these results supplement the qualitative theory by quantitative dynamic information and relations. Consequently, at least to the leading order, both qualitative and quantitative characteristics of the dynamic interaction are embodied in the flow patterns neighbouring the critical points and their distribution.

8. Concluding remarks

(i) A fluid dynamic interaction can be naturally decomposed into a normal compressing process, represented by the 'compressing variable' $\Pi = p-(\lambda+2\mu)\vartheta$, and a tangential shearing process, represented by the 'shearing variable' $\mu\omega$. This pair of variables is a pair of Stokes–Helmholtz potentials of the Navier–Stokes equation, and each has very different physical characters in its production, evolution and propagation, as well as its effect on dynamic interactions. As a complement of the primitive variables $\boldsymbol{u}$–p, fluid dynamic interactions can be alternatively interpreted in terms of $\boldsymbol{\omega}$–Π. An $\boldsymbol{\omega}$–Π interpretation of fluid/solid interaction is systematically developed in this paper. The action phase describes how the $\boldsymbol{\omega}$–Π field is created on a solid surface through the acceleration adherence (both no penetration and no slip), while the reaction phase gives the integrated force and moment acting on the solid by the $\boldsymbol{\omega}$–Π field produced. Many previously known results are included as various special cases.

(ii) In the action phase, while the decomposition splits a dynamic interaction into shearing and compressing fields, the appearance of a solid surface compels them to couple through the force balance under the adherence condition. The $\boldsymbol{\omega}$–Π coupling causes not only a boundary vorticity flux due to the tangential gradient of the normal stress Π, but also a 'compressing flux' due to the tangential gradient of $\mu\boldsymbol{\omega}$. In addition, the $\boldsymbol{\omega}$–$\boldsymbol{\omega}$ coupling, which exists only in three-dimensional flows, results in a boundary vorticity flux due to the shear stress. This $\boldsymbol{\omega}$–$\boldsymbol{\omega}$ coupling plays an important role in vorticity generation near the flow-separation region. Both the $\boldsymbol{\omega}$–Π coupling and $\boldsymbol{\omega}$–$\boldsymbol{\omega}$ coupling, as well as the effect of the body's geometry and motion, are fully described by two key boundary dynamic relationships (3.4*a*, *b*). Based on these relationships, the $\boldsymbol{\omega}$–Π production process from a solid surface is clearly identified.

(iii) In the reaction phase, the main finding was that while the local stress $\boldsymbol{t}$ is entirely determined by the boundary $\boldsymbol{\omega}$–Π (which also directly yields the total force $\boldsymbol{F}$ and moment $\boldsymbol{L}$ after integration) the real net contribution to $\boldsymbol{F}$ and $\boldsymbol{L}$ is due to the process of $\boldsymbol{\omega}$–Π being produced and fed into the fluid, a mechanism exclusively governed by boundary $\boldsymbol{\omega}$–Π fluxes. This inherent connection of the total force and moment and boundary $\boldsymbol{\omega}$–Π fluxes has not been discovered before. This connection implies that the action and reaction phases of the interaction can be combined into a unified theory.

More specifically, again due to $\boldsymbol{\omega}$–Π coupling, force and moment can always be expressed in terms of the shearing process alone. Sometimes the force can also be expressed equivalently in terms of the compressing process alone. In view of the scalar nature of this process, however, it cannot cover the full mechanism of the interaction.

This work was partially supported by the NASA Langley Research Center under the Grant NAG-1-844. The encouragement and understanding of Mr P. J. Bobbitt and Dr R. W. Barnwell was highly appreciated. We are grateful to Mr C. J. Wu, who contributed to the early development of this work during his visit to The University of Tennessee Space Institute from 1988 to 1989. We also thank Professors H. Y. Ma and T. Moulden, Mr X. H. Wu for their helpful discussions.

Appendix A. Formulae concerning tensor $\boldsymbol{B} = \vartheta \boldsymbol{I} - \nabla \boldsymbol{u}$ on a solid surface

Decompose $\nabla \boldsymbol{u}$ into symmetric and antisymmetric parts $\boldsymbol{D}$ and $\boldsymbol{\Omega}$, where $\boldsymbol{D}$ has been given by (2.2) and

$$\Omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}) = \tfrac{1}{2}\epsilon_{kij}\,\omega_k, \tag{A 1}$$

then

$$-\boldsymbol{B} = (\boldsymbol{nn} - \boldsymbol{I})\,\vartheta + \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{\omega} \times \boldsymbol{n}) + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{n})\,\boldsymbol{n} + \boldsymbol{\Omega}. \tag{A 2}$$

For simplicity a stationary wall is assumed. Then it is easy to obtain following results.

First, for any vector $\boldsymbol{f}$

$$-\boldsymbol{B} \cdot \boldsymbol{f} = -f_\pi \vartheta + \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{\omega} \times \boldsymbol{n}) \cdot \boldsymbol{f} + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{f}) - \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{f}, \tag{A 3}$$

$$-\boldsymbol{f} \cdot \boldsymbol{B} = -f_\pi \vartheta + \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{\omega} \times \boldsymbol{n}) \cdot \boldsymbol{f} + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{f}) + \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{f}. \tag{A 4}$$

In particular, if $\boldsymbol{f}=\boldsymbol{n}$,

$$\boldsymbol{B}\cdot\boldsymbol{n}=0, \tag{A 5}$$

$$\boldsymbol{n}\cdot\boldsymbol{B}=\boldsymbol{n}\times\boldsymbol{\omega}. \tag{A 6}$$

Moreover, by (A 3) and (A 6),

$$\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\boldsymbol{f})=(\boldsymbol{n}\cdot\boldsymbol{B})\boldsymbol{f}=(\boldsymbol{n}\times\boldsymbol{\omega})\cdot\boldsymbol{f}, \tag{A 7}$$

but by (A 4) and (A 5),

$$(\boldsymbol{f}\cdot\boldsymbol{B})\cdot\boldsymbol{n}=\boldsymbol{f}\cdot(\boldsymbol{B}\cdot\boldsymbol{n})=0. \tag{A 8}$$

Next, from (A 1),

$$(\boldsymbol{f}\times\boldsymbol{\Omega})_{ij}=\tfrac{1}{2}(\omega_k f_k\,\delta_{ij}-\omega_i f_j),$$

thus

$$-\boldsymbol{f}\times\boldsymbol{B}=[(\boldsymbol{f}\times\boldsymbol{n})\boldsymbol{n}-\boldsymbol{f}\times\boldsymbol{I}]\,\vartheta+\tfrac{1}{2}(\boldsymbol{f}\times\boldsymbol{n})(\boldsymbol{\omega}\times\boldsymbol{n})+\tfrac{1}{2}[\boldsymbol{f}\times(\boldsymbol{\omega}\times\boldsymbol{n})]\,\boldsymbol{n}$$
$$+\tfrac{1}{2}[(\boldsymbol{\omega}\cdot\boldsymbol{f})\boldsymbol{I}-\boldsymbol{\omega}\boldsymbol{f}]. \tag{A 9}$$

From (A 9) and (A 5) it follows that

$$(\boldsymbol{f}\times\boldsymbol{B})\cdot\boldsymbol{n}=\boldsymbol{f}\times(\boldsymbol{B}\cdot\boldsymbol{n})=0. \tag{A 10}$$

Again by (A 9),

$$-\boldsymbol{n}\times\boldsymbol{B}=-\boldsymbol{n}\times\boldsymbol{I}\vartheta+\tfrac{1}{2}[\boldsymbol{\omega}-\boldsymbol{n}(\boldsymbol{\omega}\cdot\boldsymbol{n})]\,\boldsymbol{n}+\tfrac{1}{2}[(\boldsymbol{\omega}\cdot\boldsymbol{n})\boldsymbol{I}-\boldsymbol{n}],$$

from which and with (A 3) then

$$-(\boldsymbol{n}\times\boldsymbol{B})\cdot\boldsymbol{f}=-\boldsymbol{n}\times(\boldsymbol{B}\cdot\boldsymbol{f})=(\boldsymbol{f}\times\boldsymbol{n})\,\vartheta+\tfrac{1}{2}f_\pi(\boldsymbol{\omega}\cdot\boldsymbol{n}). \tag{A 11}$$

Then, from (A 1) and (A 2),

$$-\boldsymbol{B}\times\boldsymbol{f}=[\boldsymbol{n}(\boldsymbol{n}\times\boldsymbol{f})-\boldsymbol{I}\times\boldsymbol{f}]\,\vartheta+\tfrac{1}{2}\boldsymbol{n}[(\boldsymbol{\omega}\times\boldsymbol{n})\times\boldsymbol{f}]+\tfrac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{n})(\boldsymbol{n}\times\boldsymbol{f})+\tfrac{1}{2}[(\boldsymbol{f}\cdot\boldsymbol{\omega})\boldsymbol{I}-\boldsymbol{f}\boldsymbol{\omega}],$$

hence

$$\boldsymbol{n}\cdot(\boldsymbol{B}\times\boldsymbol{f})=(\boldsymbol{n}\cdot\boldsymbol{B})\times\boldsymbol{f}=(\boldsymbol{n}\times\boldsymbol{\omega})\times\boldsymbol{f}. \tag{A 12}$$

Appendix B. Volume and surface integral identities

Let $\boldsymbol{x}$ be the position vector with arbitrary fixed origin, d the spatial dimensionality, and $k=d-1$. Then for any differentiable vector $\boldsymbol{f}$ in a domain V bounded by ∂V, the following results concerning volume integral identities occur.

COROLLARY I. *The integral of $\boldsymbol{f}$ over V can be equivalently expressed in terms of either only $\nabla\cdot\boldsymbol{f}$ or only $\nabla\times\boldsymbol{f}$, along with the boundary value of $\boldsymbol{f}$. Namely*

$$\int_V \boldsymbol{f}\,\mathrm{d}V=-\int_V \boldsymbol{x}\nabla\cdot\boldsymbol{f}\,\mathrm{d}V+\oint_{\partial V}\boldsymbol{x}\boldsymbol{n}\cdot\boldsymbol{f}\,\mathrm{d}S \tag{B 1}$$

or

$$\int_V \boldsymbol{f}\,\mathrm{d}V=\frac{1}{k}\int_V \boldsymbol{x}\times(\nabla\times\boldsymbol{f})\,\mathrm{d}V-\frac{1}{k}\oint_{\partial V}\boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{f})\,\mathrm{d}S. \tag{B 2}$$

COROLLARY II. *The integrated vectorial moment of $\boldsymbol{f}$ can be expressed solely in terms of $\nabla\times\boldsymbol{f}$, as well as the boundary value of $\boldsymbol{f}$. Namely*

$$\int_V \boldsymbol{x}\times\boldsymbol{f}\,\mathrm{d}V=\frac{1}{d}\int_V \boldsymbol{x}\times[\boldsymbol{x}\times(\nabla\times\boldsymbol{f})]\,\mathrm{d}V+\frac{1}{d}\oint_{\partial V}[(\boldsymbol{x}\times\boldsymbol{f})(\boldsymbol{x}\cdot\boldsymbol{n})-(\boldsymbol{x}\times\boldsymbol{n})(\boldsymbol{x}\cdot\boldsymbol{f})]\,\mathrm{d}S \tag{B 3a}$$

or

$$=-\frac{1}{2}\int_V x^2(\nabla\times\boldsymbol{f})\,\mathrm{d}V+\frac{1}{2}\oint_{\partial V}x^2\boldsymbol{n}\times\boldsymbol{f}\,\mathrm{d}S, \tag{B 3b}$$

and for $d = 3$ only

$$\int_V \boldsymbol{x} \times \boldsymbol{f} \, \mathrm{d}V = \int_V \boldsymbol{xx} \cdot (\nabla \times \boldsymbol{f}) \, \mathrm{d}V - \oint_{\partial V} \boldsymbol{xf} \cdot (\boldsymbol{x} \times \boldsymbol{n}) \, \mathrm{d}S. \tag{B 3c}$$

COROLLARY III. *The integrated $\boldsymbol{x} \times \boldsymbol{f}$ cannot be expressed in terms of $\nabla \cdot \boldsymbol{f}$ only.*

Note that in three-dimensional space a vector $\boldsymbol{\varphi}$ has three different second-order vectorial moments $\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\varphi})$, $x^2 \boldsymbol{\varphi}$ and $\boldsymbol{x}(\boldsymbol{x} \cdot \boldsymbol{\varphi})$, related by

$$\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\varphi}) = \boldsymbol{x}(\boldsymbol{x} \cdot \boldsymbol{\varphi}) - x^2 \boldsymbol{\varphi}. \tag{B 4}$$

When $d = 2$, for $\boldsymbol{\varphi} = \nabla \times \boldsymbol{f}$, the moment $\boldsymbol{x}(\boldsymbol{x} \cdot \boldsymbol{\varphi})$ is identically zero.

The proof of the above results is a routine exercise of vector analysis. First, (B 1) and (B 2) are direct consequences of a pair of vector identities

$$\boldsymbol{x} \nabla \cdot \boldsymbol{f} = \nabla \cdot (\boldsymbol{fx}) - \boldsymbol{f}, \tag{B 5}$$

$$\boldsymbol{x} \times (\nabla \times \boldsymbol{f}) = \nabla(\boldsymbol{x} \cdot \boldsymbol{f}) - \nabla \cdot (\boldsymbol{xf}) + k\boldsymbol{f} \tag{B 6}$$

along with the Gauss theorem. Moreover, it can be shown that

$$\boldsymbol{x} \times [\nabla(\boldsymbol{x} \cdot \boldsymbol{f}) - \nabla \cdot (\boldsymbol{xf})] = -\nabla \times (\boldsymbol{xx} \cdot \boldsymbol{f}) - \nabla \cdot (\boldsymbol{xx} \times \boldsymbol{f}) + \boldsymbol{x} \times \boldsymbol{f},$$

which, along with (B 6), yields (B 3*a*) at once. Then

$$x^2 (\nabla \times \boldsymbol{f}) = \nabla \times (x^2 \boldsymbol{f}) - 2\boldsymbol{x} \times \boldsymbol{f},$$

which gives (B 3*b*). Similarly,

$$\boldsymbol{x}[\boldsymbol{x} \cdot (\nabla \times \boldsymbol{f})] = \nabla \cdot (\boldsymbol{f} \times \boldsymbol{xx}) + \boldsymbol{x} \times \boldsymbol{f},$$

which gives (B 3*c*), for $d = 3$ only. Equation (B 3*c*) can also be obtained by using (B 3*a*, *b*) and (B 4):

$$(d-2) \int_V \boldsymbol{x} \times \boldsymbol{f} \, \mathrm{d}V = \int_V \boldsymbol{xx} \cdot (\nabla \times \boldsymbol{f}) \, \mathrm{d}V - \oint \boldsymbol{xf} \cdot (\boldsymbol{x} \times \boldsymbol{n}) \, \mathrm{d}S,$$

which leads to $0 = 0$ for $d = 2$.

On the other hand, once a vector product is made of $\boldsymbol{x}$ and (B 5), the term containing $\nabla \cdot \boldsymbol{f}$ disappears. This justifies the corollary III.

Now consider following expressions:

$$\int_V \boldsymbol{f} \, \mathrm{d}V = -\oint_{\partial V} \phi \boldsymbol{n} \, \mathrm{d}S + \oint_{\partial V} \boldsymbol{n} \times \boldsymbol{A} \, \mathrm{d}S, \tag{B 7}$$

$$\int_V \boldsymbol{x} \times \boldsymbol{f} \, \mathrm{d}V = -\oint_{\partial V} \boldsymbol{x} \times \phi \boldsymbol{n} \, \mathrm{d}S + \oint_{\partial V} \boldsymbol{x} \times (\boldsymbol{n} \times \boldsymbol{A}) \, \mathrm{d}S \tag{B 8}$$

with $\boldsymbol{A} \cdot \boldsymbol{n} = 0$ at ∂V. Of course, $-\phi \boldsymbol{n}$ and $\boldsymbol{n} \times \boldsymbol{A}$ represent an orthogonal decomposition of a vector defined at ∂V. Let $\boldsymbol{x}$, d and k be defined as before, and denote

$$\boldsymbol{\sigma}_\phi = -(\boldsymbol{n} \times \nabla)\phi, \quad \delta_A = (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{A}, \quad \boldsymbol{\sigma}_A = (\boldsymbol{n} \times \nabla) \cdot (\boldsymbol{A} \times \boldsymbol{nn}). \tag{B 9a–c}$$

Then the following results hold.

COROLLARY IV *The surface integrals of $-\phi \boldsymbol{n}$ and $\boldsymbol{n} \times \boldsymbol{A}$ in (B 7) can be expressed by the first-order moments of their respective derivatives, namely*

$$-\oint_{\partial V} \phi \boldsymbol{n} \, \mathrm{d}S = -\frac{1}{k} \oint_{\partial V} \boldsymbol{x} \times \boldsymbol{\sigma}_\phi \, \mathrm{d}S, \tag{B 10a}$$

and
$$\oint_{\partial V} \boldsymbol{n}\times A\,\mathrm{d}S = \oint_{\partial V} \boldsymbol{x}\delta_A\,\mathrm{d}S. \qquad \text{(B 10}b\text{)}$$

In particular, for $d = 3$ only, the surface integral of $\boldsymbol{n}\times A$ can also be expressed as
$$\oint_{\partial V} \boldsymbol{n}\times A\,\mathrm{d}S = -\oint_{\partial V} \boldsymbol{x}\times\boldsymbol{\sigma}_A\,\mathrm{d}S. \qquad \text{(B 11)}$$

COROLLARY V. *The surface integral of the moment of $-\phi\boldsymbol{n}$ in (B 8) can be expressed by the second-order moments of its derivative, namely*

or
$$-\oint_{\partial V} \boldsymbol{x}\times\phi\boldsymbol{n}\,\mathrm{d}S = -\frac{1}{d}\oint_{\partial V} \boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_\phi)\,\mathrm{d}S \qquad \text{(B 12}a\text{)}$$

or
$$= \frac{1}{2}\oint_{\partial V} x^2\boldsymbol{\sigma}_\phi\,\mathrm{d}S, \qquad \text{(B 12}b\text{)}$$

and for $d = 3$ only,
$$-\oint_{\partial V} \boldsymbol{x}\times\phi\boldsymbol{n}\,\mathrm{d}S = -\oint_{\partial V} \boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{\sigma}_\phi)\,\mathrm{d}S. \qquad \text{(B 12c)}$$

COROLLARY VI. *It is impossible to express the surface integral of $\boldsymbol{x}\times(\boldsymbol{n}\times A)$ in terms of δ_A*

COROLLARY VII. *However, for $d = 3$ only,*
$$\oint_{\partial V} \boldsymbol{x}\times(\boldsymbol{n}\times A)\,\mathrm{d}S = -\oint_{\partial V} \boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_A)\,\mathrm{d}S - \oint_{\partial V} (A\cdot\boldsymbol{x})\boldsymbol{n}\,\mathrm{d}S \qquad \text{(B 13}a\text{)}$$

or
$$\frac{1}{2}\oint_{\partial V} x^2\boldsymbol{\sigma}_A\,\mathrm{d}S - \oint_{\partial V} A(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S \qquad \text{(B 13}b\text{)}$$

or
$$= -\oint_{\partial V} \boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{\sigma}_A)\,\mathrm{d}S + \oint_{\partial V} \boldsymbol{n}(\boldsymbol{x}\cdot A)\,\mathrm{d}S. \qquad \text{(B 13}c\text{)}$$

The proof of (B 10)–(B 13) is outlined as follows. First, (B 10*a*, *b*) are a direct consequence of another pair of easily proved identities
$$(\boldsymbol{n}\times\nabla)\times(\phi\boldsymbol{x}) = -k\phi\boldsymbol{n} - \boldsymbol{x}\times(\boldsymbol{n}\times\nabla\phi) \qquad \text{(B 14)}$$
$$(\boldsymbol{n}\times\nabla)\cdot(A\boldsymbol{x}) = A\times\boldsymbol{n} + \boldsymbol{x}(\boldsymbol{n}\times\nabla)\cdot A, \qquad \text{(B 15)}$$
along with the Stokes theorem (3.11).

Then, by (B 9*a*) the vector product of $\boldsymbol{x}$ and (B 14) leads to
$$-k\boldsymbol{x}\times\phi\boldsymbol{n} = \boldsymbol{x}\times[(\boldsymbol{n}\times\nabla)\times(\phi\boldsymbol{x})] - \boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_\phi)$$
$$= (\boldsymbol{n}\times\nabla)\cdot[(x^2\boldsymbol{I} - \boldsymbol{x}\boldsymbol{x})\phi] + \boldsymbol{x}\times\phi\boldsymbol{n} - \boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_\phi),$$

hence
$$-d(\boldsymbol{x}\times\phi\boldsymbol{n}) = (\boldsymbol{n}\times\nabla)\cdot[(x^2\boldsymbol{I} - \boldsymbol{x}\boldsymbol{x})\phi] - \boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_\phi), \qquad \text{(B 16)}$$
from which and with (3.11) one has (B 12*a*). Moreover, it can be shown that
$$x^2(\boldsymbol{n}\times\nabla\phi) = (\boldsymbol{n}\times\nabla)(x^2\phi) + 2\boldsymbol{x}\times\phi\boldsymbol{n} \qquad \text{(B 17)}$$
from which and with (B 9*a*), (B 12*b*) follows. Then (B 16) and (B 17) give
$$(d-2)(\boldsymbol{x}\times\phi\boldsymbol{n}) = (\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{x}\boldsymbol{x}\phi) + \boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{\sigma}_\phi$$
which leads to (B 12*c*) if $d = 3$, and $0 = 0$ if $d = 2$.

In order to express the integral of $\boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{A})$ in terms of δ_A, the only starting point is (B 15). But the vector product of $\boldsymbol{x}$ and (B 15) leads to the trivial result $0=0$. Thus the result is VI.

There remains the proof of (B 11) and (B 13*a–c*). For this purpose let $\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{n}$ and rewrite (B 9*c*) as

$$\boldsymbol{\sigma}_A=(\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{B}\boldsymbol{n}). \tag{B 18}$$

It can be shown that

$$\boldsymbol{x}\times[(\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{B}\boldsymbol{n})]=(\boldsymbol{n}\times\nabla)\cdot[\boldsymbol{B}(\boldsymbol{x}\times\boldsymbol{n})]+\boldsymbol{B}-\boldsymbol{n}(\boldsymbol{B}\cdot\boldsymbol{n}).$$

Thus because $\boldsymbol{B}\cdot\boldsymbol{n}=0$ we have (B 11). Moreover, the above equality gives

$$\begin{aligned}\boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{A})=-\boldsymbol{x}\times\boldsymbol{B}&=\boldsymbol{x}\times\{(\boldsymbol{n}\times\nabla)\cdot[\boldsymbol{B}(\boldsymbol{x}\times\boldsymbol{n})]\}-\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_A)\\&=(\boldsymbol{n}\times\nabla)\cdot[\boldsymbol{B}\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{n})]-\boldsymbol{n}(\boldsymbol{A}\cdot\boldsymbol{x})-\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\sigma}_A),\end{aligned} \tag{B 19}$$

which, by (3.11), leads to (B 13*a*). Then

$$\begin{aligned}x^2\boldsymbol{\sigma}_A&=(\boldsymbol{n}\times\nabla)\cdot(x^2\boldsymbol{B}\boldsymbol{n})-2(\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{B}\boldsymbol{n}\\&=(\boldsymbol{n}\times\nabla)\cdot(x^2\boldsymbol{B}\boldsymbol{n})+2(\boldsymbol{x}\cdot\boldsymbol{A})\boldsymbol{n}-2\boldsymbol{A}(\boldsymbol{x}\cdot\boldsymbol{n})+2\boldsymbol{A}(\boldsymbol{x}\cdot\boldsymbol{n})\\&=(\boldsymbol{n}\times\nabla)\cdot(x^2\boldsymbol{B}\boldsymbol{n})+2\boldsymbol{x}(\boldsymbol{n}\times\boldsymbol{A})+2\boldsymbol{A}(\boldsymbol{x}\cdot\boldsymbol{n}),\end{aligned} \tag{B 20}$$

which gives (B 13*b*). Similarly, it can be shown that

$$\boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{\sigma}_A=(\boldsymbol{n}\times\nabla)\cdot(\boldsymbol{B}\boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{n})-\boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{A})+\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{A}) \tag{B 21}$$

which leads to (B 13*c*) and is non-trivial for $d=3$ only.

All the above calculations concerning various vector differentiations can be conveniently carried out by using the component form of Cartesian tensor analysis.

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